"Three models for operations planning in a supply chain context"

Lamas Vilches, Alejandro Nelson

Abstract

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Three Models for Operations Planning in a Supply Chain Context

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Abstract

The rapid growth of firms worldwide causes that relationships in supply chains become more complex, the competition between firms become stronger and customer ask for more demanding products and services. All these changes trigger that a firm looks for opportunities to get more from their scarce resources or to use resources that are beyond its direct control. Thus, a firm can achieve competitive advantages by setting partnerships with other supply chains in order to increase its production capacity, by adapting its operations and pricing strategies to the requirements of the market, and by differentiating between customers according to necessities and willingness to pay for its offered products.

We start by studying how firms with similar supply chain roles can collaborate in their operations. Through this collaboration, the firms can mutually benefit from the synergies of pooling operations. A successful collaboration should be efficient in reducing the global cost of the firms. Nevertheless, the willingness of each firm to collaborate may also depend on the fairness of the global cost allocation.

In order to guide joint operations, we propose a novel approach that gives priority to the firm that tends to benefit less from the collaboration. In particular, our approach balances the global cost reduction and the inequalities of allocating such cost. Moreover, when analyzing the effect of the uncertainty on the collaboration, our approach shows a high efficiency in reducing risk in comparison to other approaches.

We continue by studying how a firm can coordinate its pricing strategy and production and inventory decisions in order to compete with other firms in terms of the prices offered to the customers. We propose a framework for simultaneously planning prices and operations of a firm. We model the operations of the firms as a Lot sizing problem, so computing equilibrium between the prices becomes a time demanding task. We reduce significantly the computational time by providing bounds for the profits that firms can achieve by selecting a pricing strategy. Also, we show the convenience for the firms of using dynamic pricing strategy instead of static pricing strategy, and we show the negative effects that may arise when increasing production capacities.

We also study how a firm can gain from differentiating the demand of their customers based on profits and manufacturing requirements. Our study is motivated by the case of a firm that faces two types of demand: on the one hand there are the so-called regular orders that have a relatively long lead time; on the other hand there are urgent orders, whose delay is much shorter but their margins are higher. We study the order acceptance problem for a firm that serves two classes of demand over an infinite horizon. The firm has to decide whether to accept a regular order (or equivalently how much capacity to set aside for urgent orders) in order to maximize its profit. We formulate this problem as a multi-dimensional Markovian Decision Process. We propose a family of approximate formulations to reduce the dimension of the state space via aggregation. We show how our approach can be used to compute bounds on the profit associated with the optimal order acceptance policy. Finally, we show that the value of revenue management is commensurate with the operational flexibility of the firm.

Dedication

To Mónica and Nelson for giving me everything that a son could expect, and more.
To Tanja for her unconditional love and support.

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Alejandro Lamas
Louvain-la-neuve
December, 2014

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The main objective of a firm is to create value by utilizing its resources efficiently. Such value creation and efficiency allow the firm to maximize profits and/or minimize costs. With this in mind, the success of a firm is linked to good planning of its internal operations and actions. Moreover, a firm should take its business environment into consideration. Indeed, in the last years we have been witness of how the links between firms have strengthened as a consequence of the advances in managing information, the ability to transport larger volumes faster and the necessity of designing products across firms to satisfy the more specific customers requests. Also, the firm faces more competitive environments, which arise from the specific requirements and high standards expected by the customers together with the global proliferation of firms. Hence, the decisions of a firm are more dependent on the actions of its competitors.

In the last decades, diverse studies in Supply Chain Management (SCM) focus on modeling the relations between different firms. SCM combines two research subjects: Microeconomics and Operations Management. On the one hand, Microeconomics is the traditional perspective to study the inter-relations between firms, however, the fact that microeconomic approaches are based on simplified production functions, which constitutes a barrier for the implementation of such models in practice. On the other hand, firms organize their production activities by implementing models of Operations Management. In general, Operations Management methodologies study the operations of a firm in isolation from its business environment. Nevertheless, the recent advances in the Operations Research techniques (e.g. Mathematical Programming and Simulation) permitted to join the internal operations planning with the linking of firms.

From a SCM perspective, the interactions between the entities belonging to the same supply chain (SC) are referred to as vertical relationships. The connections between different supply chains (SCs) have been classified as horizontal relationships. Here, we describe such relationships and discuss the main challenges related to them.

1.1 Horizontal Relationships in Supply Chains

Horizontal relationships involve business interactions between entities that have similar roles in different SCs. The horizontal relationships occur between two types of entities: competitors and non competitors. To demonstrate the difference, we consider the case of two publishers belonging to different SCs that print different books or journals. If the the publishers offer their items in the same geographical region, a competitive relationship arises naturally between them, in which each publisher fights to get a larger proportion of common potential customers. Also, the publishers may compete when the raw materials (ink, paper, etc.) utilized to manufacture their items are provided by a common supplier. A case that illustrates the horizontal relationship between non competitors is when publishers offer their books or journals in different geographical regions, countries or even continents, as their potential customers constitute mutually exclusive groups. In general, this phenomenon occurs when the entities offer products that point to different market segments.

Horizontal relationships between competitors has been largely studied in the literature on Microeconomics. The goal of the microeconomic approaches is to determine the quantities and characteristics (e.g. price, quality, lead time, etc.) of the items or the services offered by different firms, that would lead to a market equilibrium. From a SCM perspective, the microeconomics approaches are reinforced by including explicitly the operations planning of the firms thus the offers of a firm are linked with the actual availability of products, the production flexibility for promising lead times, quality of the items, etc.

Regardless of the competitive or non competitive nature of the firms involved in a horizontal relationship, when entities have the same kind of roles in different SCs, there is an opportunity to exploit the synergies arising from the similarities of the activities that they carry out. The presence of such potential synergies motivates entities belonging to different SCs to pool their business activities, in such a way that they mutually benefit. This type of agreement is classified as horizontal collaboration. By horizontal collaboration the firms can increase revenues, decrease costs, expand geographical coverage, etc. For instance, if the agreement between the book publishers considers sharing the fleet of vehicles for the delivery of the items, the publishers may reduce the size of such fleet, decrease transportation costs and cover larger geographical regions. Also, the publishers can agree to pool their stocks of final products or raw materials, that may lead to a larger availability of items and a reduction of the total holding costs. Furthermore, the publishers can pool their production lines, in such a way that they gain from the economies of scales arising from reducing fixed productions costs and increasing the flexibility of the manufacturing activities. In addition, when the operations of the publishers are subject to uncertainty, pooling operations can bring benefits in terms of reducing the risk of the firms (the well known risk pooling effect).

Even though the entities that decide to collaborate may be non competitors, collaboration has a competitive dimension born of the conflicting objectives of the partners. In our example, even when the two publishers offer their items in different geographical regions, they will compete for the utilization of the shared resources if they decide to pool their production lines. The biggest challenge in horizontal collaboration is to address simultaneously the collaboration and the competition between partners. Indeed, the success of horizontal collaboration depends on how the agreement ensures the efficiency of the shared actions while taking into account the interests of each partner.

1.2 Vertical Relationships in Supply Chains

Vertical relationships cover the traditional linkages between entities belonging to the same SC, i.e. the interactions between suppliers and customers. Although, creating value by offering the right products or services to a final customers is a common objective across a SC, the different echelons of the SC may have conflicting objectives. Addressing such conflicting objectives constitutes one of the main interests of the literature on SCM. An example of conflicting objectives is the lead time of the orders that a retailer demands from a manufacturer. Shorter lead times will impact positively on the operations of the retailer (e.g. reduction of stocks, operations flexibility), while the manufacturer will face more restricted operations. Another example is the price that a manufacturer demands for the items that it sells to the retailer. Higher prices may increase the profit of the manufacturer, while they impact negatively on the profit of the retailer.

A manufacturer should make two important decisions that may affect its relations with retailers. First, the manufacturer should make decisions that exploit the particularities of the demands of the retailers. Specifically, the demands may differ in terms of value, urgency, quality, etc., hence the manufacturer can capture larger surpluses by offering items adapted to the expectations of each retailer. consider the case of a publisher (manufacturer) that receives two types of orders: best-sellers, which are characterized by a high rotation, and regular books, for which orders are less demanding in terms of delivery times and volumes. The publisher can ensure quick delivery for the best-sellers, but given the high demand for best-sellers at bookstores, the willingness of bookstores to pay higher prices for quick delivery of such books is higher than for the regular books. Thus, the publisher can charge a premium for the best-sellers. Second, the manufacturer should fulfill its agreement with the retailers ensure that the expectations of the retailers are effectively met. Once the manufacturer defines its offer, its priority should be to meet the retailer expectations. This results in stronger loyalty of the retailers, and consequently, it may lead to a competitive advantage for the manufacturer. Thus, the publisher plans its operations in order to maximize its revenue while taking into account the promised quantity and delivery times agreed with a bookstore.

1.3 Scope of the Thesis

Taking an SCM perspective, the main objective of this thesis is to study how firms should plan their operations when taking into account the relations with agents of its own supply chain or entities belonging to other SCs. We focus on operations planning based on the following observations:

- Despite the progress in the subject of SCM, there still exist gaps between practice and the theoretical models for addressing the interaction among firms. For example, the benefits of collaboration are largely discussed in the literature, but little research focuses on how to implement horizontal agreements in practice. We expect to bring the collaboration to real environments through providing frameworks that help firms to simultaneously plan their operations in collaborative initiatives. We observe the same gap in competitive environments. Many works address the relationship of competition from a pure microeconomics point of view, but few of them model in detail the operations of the firms.

- Even though a well structured relationship at the strategic level or at the tactical level may show a clear equilibrium between firms in the long term, the implementation and evolution of a decision will depend on the results of a firm in the short term. For instance, a group of firms can ensure the reduction of its operational costs by working together in the long term, but if each firm cannot perceive reduced costs in the short term, the willingness to continue working together may be weak.

- A firm should coordinate the decisions of its different departments in its operations planning. For example, the marketing department decides the prices and quantities of items in order to compete with other firms, but verifying the costs of making available such offer requires coordination with the operations department. So, making an offer has to be studied by both departments.

Our goal is to lessen this gap by providing models that help firms to plan their operations in a collaborative environment, both competitive and non-competitive. Since we focus on the operations of the firms, we utilize quantitative approaches based on Operations Management and Operations Research, such as Mathematical Programming, Markov Decision Process (MDP) and Simulation. The results of our analytical models allow us to propose managerial insights that can enlighten and promote the collaborative actions between firms. Further, our results can be used by managers and researchers in order to understand the effects of competition on the operations of a firm. Also, we show how a firm can increase its revenues by differentiating between the expectations in terms of price and delivery time of its customers.

This thesis is divided into three parts, each of them covering a different aspect of the vertical and horizontal relationships in SCs. In the first part, our objective is to provide a general framework for pooling operations between firms. We propose analytical models for implementing horizontal collaboration between different SCs. The second part addresses the competitive relationships between SCs. Specifically, our aim is to provide an analytical model that addresses simultaneously the pricing decisions and the operational planning. In the third part, our goal is to provide algorithms for solving the problem of a manufacturer that must decide whether to accept or reject orders from retailers. Explicitly we consider a vertical relationship in which a manufacturer offers different lead times and prices to the retailers.

1.3.1 Fairness in Operations Pooling

In this part of the thesis, we focus on horizontal collaboration initiatives in which two firms pool their operations. These firms play similar roles in different SCs, and they point to markets that are mutually exclusive (non competitors).

The motivation of this study is born from the fact that independent firms may benefit from collaborative alliances by reducing their costs and risks. Particularly in operations, different firms can gain from economies of scale by pooling their production resources. We summarize the core idea of this part of the thesis in Figure 1.1.

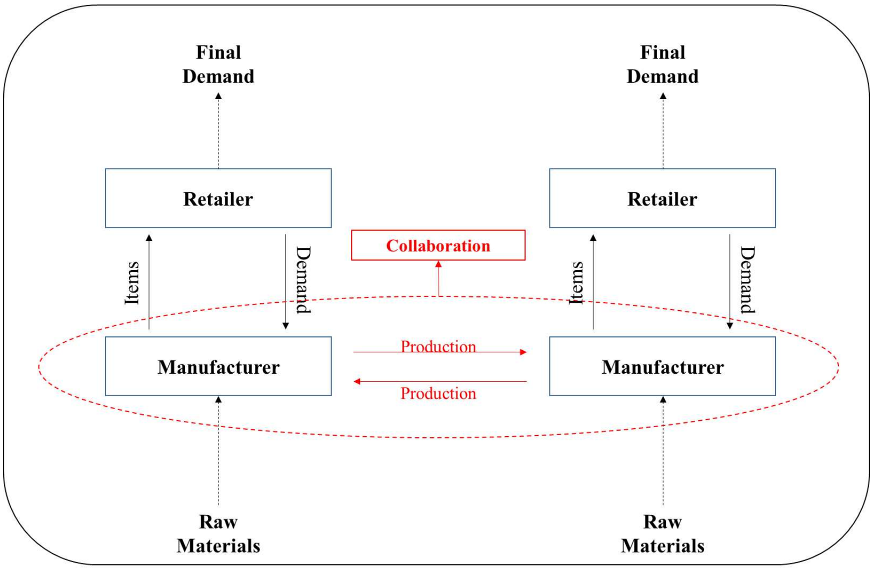


Figure 1.1: *Painess in Operations Pooling.*

Even though there may be a significant reduction of the overall cost, the success of a partnership may depend on the fairness of the agreement. With this in mind, the firms can utilize transfer payments to balance the benefit allocation. The implementation of such payments, however, could be difficult in practice, because of the presence of legal constraints or additional contracts that make such agreements more complicated. Therefore, the partners should jointly plan operations taking into account the trade-off between efficiency and fairness.

In the problem that we study, we model the operations of the firms as a Capacitated Lot Sizing Problem (CLSP). The firms need to agree on a collaboration scheme before the operations are carried out. Once demands are realized, the CLSP is used for jointly planning the operations of the firms.

We propose a novel methodology to guide joint operations based on Rawls' theory of justice, in which we prioritize improving the firm that tends to benefit less from the collaboration. We show that our proposed methodology is effective in reducing the inequalities in the results of the partners while the efficiency is not significantly affected. Moreover, when firms make the agreement under uncertainty, our methodology has a positive impact on reducing the risk of the firms. In particular, our proposed methodology outperforms other collaboration schemes in terms of balancing risk, efficiency and fairness.

1.3.2 Pricing and Operations Planning Under Competition

Here, we study the horizontal relationship between two firms that produce items which are offered to the same retailers or customers (see Figure1.2). The motivation for this study lies in the fact that the pricing decisions that a firm makes with the aim to compete with its opponents are mutually linked to the operations planning of the firm and its opponents.

Again, we assume that the operations of the firms are modeled as a Lot Sizing Problem (LSP). In terms of the pricing decisions of the firms, we assume that each firm sets a unit selling price for each period of production, and there is a discrete set of prices that a firm can choose from for each period. Furthermore, we assume that the pricing decisions can influence the demands, i.e. demands are price sensitive.

In contrast to the problem studied for horizontal collaboration agreements, here each firm makes its own operations planning. Nevertheless, such planning should take into account its own pricing decisions and the prices chosen by its opponent. Hence, the pricing decisions and the operations planning of the firms correspond to a Nash Equilibrium (NE).

Given that the LSP is a discrete problem and each firm chooses prices from discrete sets, the computational effort required to calculate a NE may be large when the planning horizon reaches a certain length. We propose a framework consisting of Mixed Integer Programming formulations that provides a NE. This framework reduces the computational time by characterizing pricing strategies that firms will potentially use at equilibrium.

Our numerical results provide new managerial insights about the sensitivity of the pricing decisions to changes in the characteristics of the firms, such as the level of

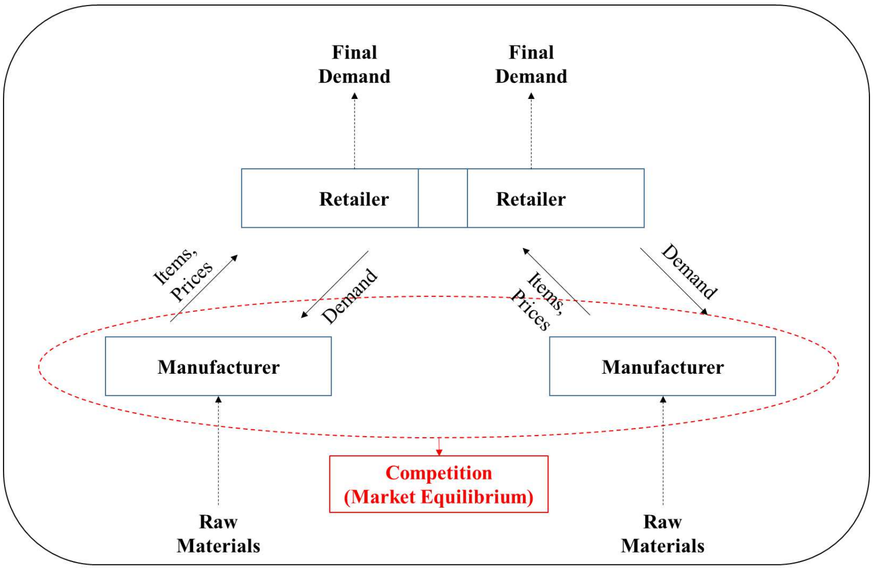


Figure 1.2: *Pricing and Operations Planning Under Competition.*

substitution between their items and the seasonal effects of the demands. When the operations of the firms have capacity constraints, i.e. when we model the operations as a CLSP, increasing the production capacity may be self-defeating for the firms.

1.3.3 Planning Operations for Revenue Maximization Under Lead Time Constraints

In this part of the thesis we study how the operations of a firm should be adapted to the prices and lead times promised downstream in the SC. In particular, our study is motivated by the case of a firm manufacturing industrial equipment that faces two types of demand. On the one hand there are the so-called regular orders for installation or refurbishing of existing facilities, these orders have a relatively long lead time. On the other hand there are urgent orders mostly related to spare parts required when a facility has a breakdown. The delay in such a case is much shorter but higher margins can be obtained.

We study the order acceptance problem for a firm that serves two classes of demand over an infinite horizon. The firm has to decide whether to accept a regular order (or equivalently how much capacity to set aside for urgent orders) in order to maximize its profit, as we show in Figure 1.3. We formulate this problem as a multi-dimensional MDP.

Given that solving the problem through the MDP formulation may require a large computational effort, we propose a family of approximate formulations to reduce the

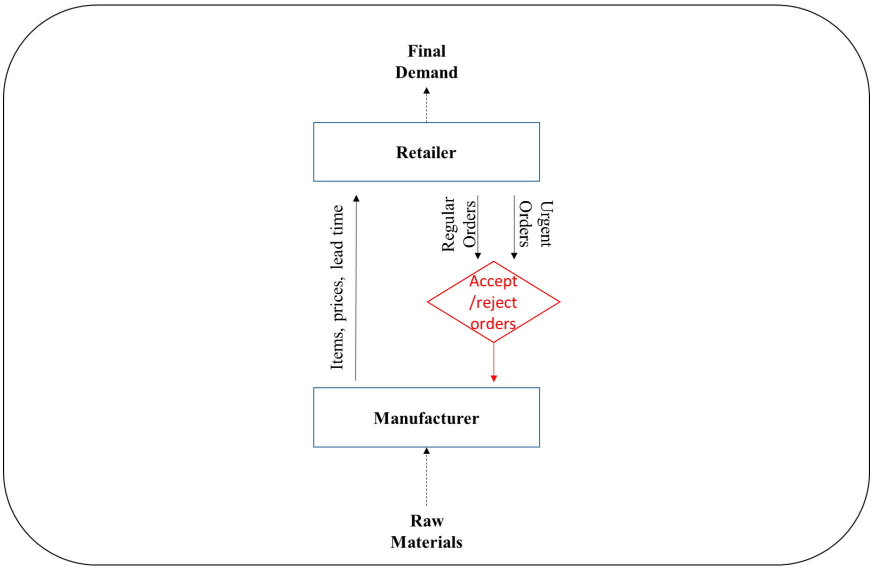


Figure 1.3: *Planning Operations for Revenue Maximization Under Lead Time Constraints.*

dimension of the state space via aggregation. Moreover, the aggregated formulation provides bounds for the profit associated with the optimal order acceptance policy.

1.4 Outline of the Thesis

In Chapter 2 we study the operations pooling between SCs. We give special attention to the definition and characterization of the fairness of a collaboration agreement, which is the key for the success of the collaboration. Chapter 2 is based on Lamas and Chevalier (2013).

In Chapter 3 we continue studying horizontal relationships between SCs, but now we focus on the competition between different SCs. Here, we explain the linkage between the pricing decisions and the operations planning of the firms in competitive environments.

In Chapter 4, we describe the operations planning of a manufacturer that aims to maximize revenues when there are lead time constraints. The main focus of the chapter is to provide heuristic methods for making capacity allocation decisions quickly. This chapter is based on Chevalier et al. (2015).

Finally in Chapter 5 we present the main conclusions and contribution of this thesis. Also, we provide some ideas for the extensions of this thesis for future research.

Chapter 2

Fairness in Operations Pooling

2.1 Introduction

This work is motivated by the operations of two manufacturers that produce LCD panels. As the panel types have similar technical specifications, the manufacturers can gain from the economies of scale that arise from utilizing interchangeably their production lines. For instance, if a line has slack capacity in a production period, the full production of another firm in this period can be allocated to this line, so only one of the lines is active, and the manufacturers reduce their total set-up costs. Further, a manufacturer can utilize the capacity slack of another line to produce panels as late (early) as possible, in such way its total holding (backordering) cost is reduced.

Although pooling operations may bring significant gains to the manufacturers, there exist some impediments for setting up this initiative. Such impediments are related to the conflicting objectives between the manufacturers. We can say that a stress between partners arises naturally from operations pooling, because once the manufacturers decide to pool their operations, they will compete for utilizing the common production resources. So, the manufacturers should reach a consensus on how to jointly plan their operations, otherwise, the success of the operations pooling may be in jeopardy. The consensus has to be materialized as an agreement between manufacturers in which the allocation of resources takes the individual interests of each partner into account.

Given that future partners will share the benefits of the operations pooling, larger benefits of the association increase the attractiveness of sharing. In other words, as setting up the operations pooling requires a significant effort by the future partners, larger global gains make it more likely to surpass such efforts. Hence, when setting up the agreement, one of the essential challenges is to answer the question: 'How do we expand the pie of benefits between collaborative parties?'(Jap, 1999). We denominate efficiency as the measure of the overall benefit achieved through pooling operations.

Another aspect that partners should address in their agreement is the allocation of the gains. Reaching a consensus on how to divide the gains is crucial for preventing conflicts between partners. In this sense, the future partners expect to reach an agreement in which the allocation of gains is fair. The notion of fairness may be the key element of the success of the operations pooling. Although the relevant role of fairness in alliances is widely discussed in the literature, there are multiple definitions for such notion, see e.g. Nash (1950a), Rawls (1971), Kalai and Smorodinsky (1975), Camerer and Thaler (2010), Bertsimas et al. (2011). Several behavioral experiments illustrated that partners might prefer a self-destructing behavior to unfair agreements. The partners can desist from working together, even though each of them could improve its performance compared to the stand-alone situation. An example is the experiment of the ultimatum game, in which a proposer offers some allocation of money between himself and a responder, who can decide to accept it or not. Contrary to the expected rational behavior, if the offered proportion is strictly positive, but smaller than a certain value, the responder tends to reject the offer (Camerer and Thaler, 2010, Jap, 2001). Motivated by this behavior, recent literature shows that the expectations of participants to economic relations are complex, and they also involve the relative results between firms.

Maximizing efficiency while meeting fairness is a challenging goal for the agreement between future partners. The difficulty to reach such goal is impacted by the existence or not of transfer payments between partners. If utility can be transferred between partners through monetary payments, the feasibility of the maximum efficiency is independent of the allocation of gains. Indeed, the partners can always implement the decision that leads the maximum efficiency, and then determine the adequate payments that meet the fairness criteria. Nevertheless, firms are generally reluctant to transfer payments, because the collaboration takes into account several intangible costs (for example, holding and back-ordering costs) which are difficult to evaluate even when the production plan is carried out in isolation. Moreover, the requirements of contracts between partners and/or the presence of legal restrictions are additional impediments for the implementation of transfer payments. This has been widely reported in the literature, see e.g. Granot and Sosic (2005), Makadok (2010). In the absence of such payments, implementing the globally optimal operations may be incompatible with the fairness conditions established in the agreement. Thus a trade-off exists between the fairness and the efficiency dimensions of the agreement. Given that the operations will directly determine the shares of the benefits when payments are not possible, the management of operations is crucial for dealing with such trade-off.

Further, the agreement should also address the fact that the partners must be sufficiently confident in advance in the convenience of the operations pooling. Thus, one more aspect that may influence the success of the operations pooling is the risk of the agreement. This risk arises from the uncertainty faced by the firms when making the agreement. For example, the manufacturers of LCD panels will set the rules for their operations pooling before they have a full information of their future demands. So, although manufacturers are interested in minimizing their costs, each of them expects also to control the variability of its cost. Thus, controlling the risk resulting from the operations pooling helps to prevent conflicts between partners.

Given the conditions that facilitate a consensus on how to pool operations: fairness, risk reduction and absence of transfer payments, we propose a methodology for jointly planning operations in order to overcome such hurdles. Our methodology is characterized as follows. On the one hand, we model the production line of each manufacturer as a Capacitated Lot Sizing problem (*CLSP*), a well-known production-inventory problem that establishes when and how much to produce in order to minimize set-up, production, holding and back-ordering costs (see Wolsey (1995), Karimi et al. (2003), Brahimi et al. (2006) for an extended literature review of this problem). The formulation of the *CLSP* is consistent with the practical case that motivates our work, where the manufacturers can gain from the economies of scale resulting from pooling operations. On the other hand, we address the conflicting objectives of the manufacturers by proposing agreements inspired on the Rawlsian notion of fairness, i.e., the agreement between partners prioritizes the improvement the partner that is least well off (Rawls, 1971). Moreover, we measure the effect of the proposed methodology on the risk of each firm in comparison to other types of agreements.

Our work contributes to the scientific discussion by providing a methodology for operations pooling that combines games or group decisions, multi-criteria approaches and production-inventory problems. In general, the traditional approaches for operations pooling assume that the willingness to participate on a partnership initiative depends solely on the individual rationality of the firms (i.e. the operations pooling does not hurt the performance of any partner). Although some approaches assume that partners may prefer fair agreements, those approaches also assume the possibility of transfer payments between partners. We extend traditional approaches for pooling operations by both including a sophisticated notion of fairness and lifting the transfer payments assumption. We also provide a novel analysis for the link between fairness and risk of the operations. Therefore, our methodology corresponds better to the interest of firms in real applications.

From a managerial point of view, this research may inspire the implementation of operations pooling in real environments, because the proposed methodology helps the firms to jump hurdles for collaboration (e.g. fairness, risk or absence of transfer payments). So, we contribute to shorten the gap between the theory of operations pooling model and the implementation of collaborative practices in practice.

The Chapter is structured as follows. In Section 2.2 we highlight the literature related to our research. Section 2.3 describes the problem of firms when they jointly plan their operations. In Section 2.4 we present and discuss the measure and criteria for the efficiency, the fairness and the risk of a collaborative agreement. In Section 2.5 we introduce schemes leading to decisions that make a balance between the different dimensions of a collaboration problem. We introduce our numerical analysis in Section 2.6 and we present the main conclusions of this Chapter in Section 2.7.

2.2 Literature Review

The initiative of pooling operations between firms corresponds to what has been classified as a horizontal collaboration. In general, horizontal collaborations establish joint activities between different supply chains, enabling similar parties to collaborate on a particular business function (Stadtler, 2009, Erhum and Keskinocak, 2007). There is another type of collaboration in supply chains, the vertical collaboration, which focuses on a single supply chain and it is out of the scope of our work. In practice, horizontal collaborations may be advantageous to the partners, however there are difficulties in implementing it.

We start by discussing the advantages of the horizontal collaboration. Corbett and de Groote (2000) and Simatupang and Sridharan (2002) suggest that the essential advantages lie in the improvements in comparison to an initial solution (stand-alone situation). According to Crujissen et al. (2007), such improvements arise from reducing the costs of operations and increasing the productivity of the firms. Note that, there may be other advantages such as improving the service level (lead times, geographical coverage, reliability, etc.) and better competitive position of the firms in their markets, however, these are in general secondary objectives to the cost reduction. For such reason, our study focuses on the effect of collaboration on the costs of the firms.

We continue by describing the impediments for implementing horizontal collaborations. The main difficulty in implementing such practice is the division of the gains (Crujissen et al., 2007). Brandenburgger and Nalebuff (1996) emphasize that collaborations should prioritize the maximization of the overall benefit of the companies, but it is essential to take into account that the different parties will compete for such gains. To handle the stress resulting from such competition, the partners must perceive that the collaborative agreement is fair. Indeed, Bertsimas et al. (2011) suggest that in many environments fairness is more important than optimality. Further, the authors propose that when different parties collaborate, a decision may not be practically implementable, because some of the parties might consider it 'unfair'.

Although the relevance of fairness is clear in collaboration, the meaning of such concept is ambiguous and vastly discussed in the literature. Nevertheless, most of the existing notions of fairness compare the relative results achieved by the partners. In this line, Camerer and Thaler (2010) suggest that participants do not care about the other's welfare per se, but desire some type of equity. Thus, participants may even prefer self destructing behaviour to unfair collaborations. In order to implement such notion of fairness, Fishburn and Sarin (1994), Boiney (1995), Brams and Taylor (1995) utilize the concept of equity, which establishes that each partner must receive the same proportion of the gain resulting from collaboration. Alternatively, many authors follow the Rawlsian notion of fairness (Rawls, 1971), in which the collaboration focuses efforts to the party that is least well off. With the aim of establishing common criteria for collaborative agreements, Nash (1950a) proposes an axiomatic characterization of fairness, which is extended to more general problems by Kalai and Smorodinsky (1975) and Kalai (1977). Note that, the cost allocations derived from the axiomatic characterization of Kalai and Smorodinsky (1975) and Kalai (1977) is of the Rawlsian type. However, no notion in the literature satisfies simultaneously the whole set of fairness axioms. Any collaborative initiative should, therefore, be based on a notion that covers the maximum number of axioms. In this work, we are devoted of the axiomatic characterization proposed Kalai and Smorodinsky (1975) and Kalai (1977), because the Rawlsian notion of fairness is consistent with the expectations of partners in business environments (Kumar, 1996).

The existing formulations for addressing fairness in collaboration can be divided into two categories depending on the existence, or not, of transfer payments between the firms. When such payments are possible, the problem of collaboration can be divided in two stages: first, the firms make decisions in order to achieve global optimality; based on such decisions, the resulting benefits are divided between firms. Note that, the globally optimal decision is always feasible, since transfer payments lead to an equilibrium between the interests of the different parties. The problem of collaboration with transfer payments is largely studied in the literature of cooperative games with transferable utility (TU-game), see e.g. von Neumann and Morgenstern (1943), Shapley (1959), Gillies (1959), Myerson (1991). In the same line, but more related to our work, Fiestras-Janeiro et al. (2011) review the most relevant approaches dealing with collaboration in the context of production-inventory problems. A first group of such approaches focuses on the case where demand is continuous and constant, so that firms collaborate in an economic order quantity (EOQ) environment. Meca et al. (2004), Federgruen and Zheng (1992), Anily and Haviv (2007) deal with the EOQ problem with multiple firms. The authors propose a formulation, in which once the firms minimize their joint inventory cost, the costs are allocated between firms by utilizing a game theory approach. Another group of approaches focuses on dynamic demand problems, whose game theory formulations are known as production-inventory games. Guardiola et al. (2009) deal with this type of problem, but in contrast to our case, their problem ignores the fixed ordering costs, so the analysis omits the economies of scale resulting from collaboration. van den Heuvel et al. (2007) study what they call Economic-lot sizing game, in which the operations of the firms involve fixed ordering costs. The authors model the operations as an Uncapcitated Lot-sizing problem, but the game associated to the cost allocation between firms has a non-linear cost structure caused by the presence of such costs. Sambasivan and Yahya (2005), Drechsel (2010), Drechsel and Kimms (2011) extend the problem to its capacitated version. Sambasivan and Yahya (2005) proposes a Lagrangean relaxation based heuristic to solve the problem, but the authors do not address the allocation of costs between firms. Drechsel (2010) addresses the allocation problem through a max-min formulation such that the difference between the maximum relative reduction of costs of the firms is minimized. This work is in many aspects close to this current investigation, but there are two essential differences: (i) their formulation considers transfer payments between firms, and (ii) their performance measurement is different.

There are practical difficulties related to collaborations that arise in both calculating the exact benefits of each firm (consequently difficulties on determining the amount of side payments) and possible legal restrictions (Granot and Sosic, 2005). Moreover, when two competitors decide to collaborate, their transfer payments are likely to be reviewed by the antitrust authority. Thus the absence of such payments facilitates such agreements. The problem of collaboration in the absence of payments is studied in the literature as non-transferable utility games (NTU-game), see e.g. Aumann (1961), Myerson (1991), Borm et al. (1992). The main difference in comparison to the TU-games lies in the fact that the operational decisions of the firms directly determine the allocation of gains. Then, if the firms make agreements taking the fairness dimension into account, the efficiency of the collaboration in the NTU-games is potentially reduced in comparison to the TU-games (Jain and Mahdian, 2007). Bertsimas et al. (2011) study the inefficiency in NTU-games when fairness considerations are introduced. The authors propose upper bounds for such inefficiency under different notions of fairness. However, the bounds are only valid for convex utility sets, we try in this article to determine decision rules that implement different notion of fairness in the framework of production planning where the utility set is typically not convex. In the same vein, Drechsel (2010), Frisk et al. (2010) propose a cost allocation methodology for the operations of firms based on the Rawlsian notion of fairness. In their case the absence of transfer payments has only a limited impact on the efficiency of the decision, because they use a linear model for operations.

As we discuss in Section 2.1, the collaboration may have an effect on the risk of the operations. Similar to fairness, risk is a concept widely discussed in the literature, but it also has multiple definitions. One of the most utilized measures of risk is the value at risk, however, Artzner et al. (1999) show the weakness of such measure. Alternatively, the authors introduce an axiomatic characterization for risk measures. The risk measures satisfying such axioms are classified as coherent. Rockafellar and Uryasev (2000) introduce a coherent risk measure known as conditional value at risk. Such measure has become common in the literature of risk and in real life applications, therefore, we utilize it for measuring the risk of the collaboration between firms.

Our work contributes to the scarce literature on horizontal collaboration in operations by proposing a methodology that prioritizes the fairness of the agreement, whose implementability is independent of the existence of transfer payments between firms. Furthermore, although there are operations management approaches for controlling the conditional value at risk (Ahmed et al., 2007, Choi and Ruszczynski, 2008, Chen et al., 2009), to the best of our knowledge, our work represents the first study including risk in operations when multiple firms pool their production resources.

2.3 Problem Description

Consider two self-interested and risk-averse firms, Firm 1 and Firm 2, that can operate interchangeably their production lines. The firms aim to jointly plan their production lines in order to take advantages of potential costs savings. Thus, the firms must agree on the rules that will govern their joint operations.

When setting up the agreement, the firms spend time and resources in the bargaining process, building trust, adapting operations, etc. Therefore, if the advantages related to the agreement do not remain valid in the long run, such effort may not be worthwhile. Consequently, we focus on long-term agreements, where the firms learn gradually their demands for a certain interval of time, we will call this interval a planning block. Thereby, firms plan operations repeatedly as reliable information of demand becomes available for a planning block. Thus, the decision in each planning block depends on the realization of demand for that interval. We say that $D \in \mathbf{R}^{2\times T}$ represents the random variables of the demands faced by the firms in each planning block, where T is the number of periods in a planning block. Also, we say that d is a realization of D.

The set of rules upon which the firms will jointly plan operations for a planning block will be called a collaboration scheme. Once the firms carry out the planned operations for a production block, they make a new joint plan for the next block based on the same scheme. Note that, determining a collaboration scheme is the main decision involved in the agreement.

In the reminder of this section, we explain the different aspects of the problem of setting a collaboration scheme.

2.3.1 Operations

We model the operations of each line during a planning block as a *CLSP*. Thus, there is a holding cost for units produced in advance of demand and a back-order cost for units produced late. Also, there are fixed production costs each period a production line is used and a unit production cost. Finally, we suppose that demand is known with enough accuracy to build a reliable production plan for a planning block consisting of T periods in advance. The parameters of the problem are $d, f, p, h, b, CAP \in \mathbf{R}^{2\times T}$, where

$d_{i,t}$: demand faced by firm i for period t

$f_{i,t}$: fixed production cost for the production line of firm i in period t

$p_{i,t}$: unit production cost of firms i's line in period t

$h_{i,t}$: unit storage cost of firm i in period t

$b_{i,t}$: unit back-ordering cost of firm i in period t

$CAP_{i,t}$: production capacity of firm i's line in period t

The decision variables of the problem are $x \in \mathbf{R}^{2\times 2\times T}$, $s, w \in \mathbf{R}^{2\times T}$ and $y \in \{0,1\}^{2\times T}$, where

$x_{i,j,t}$: amount produced on firm j's line in period t, which is destined to firm i.

$y_{i,t}$: 1 if firm i's production line is in use during period t; 0 otherwise.

$s_{i,t}$: stock level of firm i at the end of period t.

$w_{i,t}$: back-ordered units of firm i at the end of period t.

We can model the joint operations of the production lines in a planning block by the following constraints.

$$s_{i,t-1} + x_{i,1,t} + x_{i,2,t} + w_{i,t} = d_{i,t} + s_{i,t} + w_{i,t-1} \qquad \forall t = 1,..,T; i = 1,2 \tag{2.1}$$
$$x_{1,i} + x_{2,i} \leq CAP_i \cdot y_i^T \qquad \forall i = 1,2 \tag{2.2}$$
$$x, w, s \geq 0, y \in \{0,1\}. \tag{2.3}$$

Constraint (2.1) establishes the product flow conservation and constraint (2.2) is the capacity restriction in each period.

In the rest of the work, we denote by $\mathscr{X}_d$ to the set that contains all the planning decisions x that satisfy constraints (2.1) - (2.3) when the demand for a planning block corresponds to d.

2.3.2 Cost allocation

Given that the firms are self-interested, they should agree on a collaboration scheme that addresses their individual objectives. In this sense, the imputation of costs plays an important role in planning the operations pooling. Moreover, the aim of the agreement we are studying is to pool the production activities while keeping the firms as independent as possible. We reflect this in our model through the following assumptions:

- each firm supports the holding and back-ordering costs for its products,
- the fixed production costs are supported by the owner of the production line, independently of the firm for whom the products are made,

- for the unit production costs we consider two cases, either these costs are supported by the owner of the line, independently of the product, or this cost is supported by the firm for whom the product is made. The first case would correspond to a situation where the production costs are difficult to isolate (workforce drawn from the common pool of the plant, utilities where it might not be easy to determine the real cost of steam, energy, etc.). The second case will be more adequate if the direct production costs for the jointly operated line are easily determined. The general idea is that our methodology is flexible in terms of the firms' imputation of unit production costs.

Based on our cost imputation assumptions, the cost of Firm i is $f_i^T \cdot y_i + p_i^T \cdot (x_{i,1} + x_{i,2}) + h_i^T \cdot s_i + b_i^T \cdot w_i$, when the unit production costs are supported by the owner of the line, and $f_i^T \cdot y_i + p_i^T \cdot x_{i,i} + p_j^T \cdot x_{i,j} + h_i^T \cdot s_i + b_i^T \cdot w_i$, when the unit production costs supported by the firm for whom the product is made. In both cases, we say that the function $C_d(x)$ provides the pair $(C_{d,1}(x), C_{d,2}(x))$, which corresponds to the cost allocated to each firm when $x \in \mathscr{X}_d$ is implemented and the demand is d, (note that, the variables y, s and u are not required for computing the costs of the firms, because we can deduce them from x and d). In the rest of the work, we utilize $\mathscr{C}_d \in \mathbf{R}^2$ to denote the set of all cost allocations that can be obtained from the set $\mathscr{X}_d$, when the demand in the planning block corresponds to d, i.e. , $\mathscr{C}_d = \{(C_1, C_2) : C_1 = C_{d,1}(x) \wedge C_2 = C_{d,2}(x) \wedge x \in \mathscr{X}_d\}$

2.3.3 Bargaining

As we mentioned earlier, the firms should decide a collaboration scheme that will rule their joint operations. Given that future partners will compete for using their shared resources, there may be a conflict between the individual objectives of the firms. So, the success of the operations pooling depends on how the collaboration scheme agreed by the firms balances these objectives.

The basic motivation of firms when entering to a partnership is to improve its situation with respect to working in isolation. So, the firms should agree on a scheme that minimizes the joint costs of the firms. We call this dimension of the operations pooling efficiency.

Nevertheless, the conflict in operations pooling may depend on other factors. We explain this by an example. Let us assume that the firms can choose between three collaboration schemes. The common costs resulting from implementing such schemes are \$100, \$70 and \$60. At first sight, the third scheme seems to be the most interesting for the firms. However, each firm will be more interested in the scheme that minimizes its own cost, which may not be the third scheme. Let us assume that the allocation of costs is \$45 for Firm 1 and \$55 for Firm 2 in the first scheme, \$60 for Firm 1 and \$10 for Firm 2 for the second scheme, and \$10 for Firm 1 and \$50 for Firm

2 for the third scheme. Given such allocations, Firm 1 may be more interested in implementing the third scheme, but Firm 2 may be more interested in the second scheme. Moreover, implementing scheme 2 and scheme 3 are more efficient than scheme 1, but the allocation of costs is unbalanced between between firms, this could make a firm to dismiss the option of pooling operations.

So, another dimension of operations pooling is related to the apprehensions of the firms with respect to the allocation of costs. The firms should agree on a collaboration scheme with a consensus on such allocation. This consensus will mean that each firm perceives the allocation of costs as fair. This dimension of the problem is what we call fairness.

Another aspect of operations pooling is uncertainty. As the firms agree on a scheme with no certainty about their future demands, the operations of each firms are subject to risk. Given that the firms are risk-averse, each of them expects to agree on a collaboration scheme that controls its risk derived from jointly planning operations.

In the next section, we propose how to model the aspects of the bargaining problem associated to the problem of jointly planning operations.

2.4 Evaluation Criteria for a Collaboration Scheme

Aiming to address the conflicting objectives of future partners, we model the problem of agreeing on a collaboration scheme as a bargaining problem. In order to characterize the problem we utilize the following definition:

Definition 2.1. *Given the realization of demand d, the set $\mathscr{X}(d)$ and the function of costs $C_d(x)$, let $C_d^o \in \mathbf{R}^2$ be the minimum costs that firms can achieve by working separately, i.e, $C_{d,i}^o = \min\{C_{d,i}(x) \mid x_{ij} = 0, \forall i \neq j \wedge x \in \mathscr{X}(d)\}$.*

We characterize the bargaining problem by the pair $(\mathscr{C}_d, C_d^o)$, i.e. given the realized demand for the planning block, the firms make decisions considering their individual cost functions and the feasible solutions for such demand (which are implicit in $\mathscr{C}_d$), and the opportunity cost of working separately. Based on this characterization, we give the following definition for a collaboration scheme:

Definition 2.2. *Given the bargaining problem $(\mathscr{C}_d, C_d^o)$, let $C^F(\mathscr{C}_d, C_d^o) \in \mathbf{R}^2$ be the allocation of costs derived from implementing collaboration scheme F.*

In the reminder of this section, we discuss the measures and criteria for determining the efficiency, the fairness and the risk related to cost incurred by the partners in a collaboration scheme to pool operations.

2.4.1 Efficiency Measures

Before discussing our proposed efficiency measures, let us introduce a definition to facilitate the presentation of our propositions.

Definition 2.3. *Given the realization of demand d, the set $\mathscr{X}(d)$ and the function of costs $C_d(x)$, let $C_d^* \in \mathbf{R}$ be the minimum joint cost that firms can achieve by pooling operations, i.e., $C_d^* = \min\{C_{d,1}(x) + C_{d,2}(x) \mid x \in \mathscr{X}_d\}$.*

On the one hand, C_d^* constitutes a benchmark for the joint costs derived from implementing a collaboration scheme, because it corresponds to the minimum of the sum of the costs of the firms when pooling operations during a planning block when the realized demand is d. On the other hand, C_d^o represents the opportunity cost of the operations pooling, because if the firms discard the operations pooling, then they carry out their production by working separately. Based on such indicators, a natural benchmark for measuring the efficiency of collaboration scheme F is as follows:

$$\varepsilon^F = E_D \left[\frac{C^F(\mathscr{C}_d, C_d^o) - (C_{d,1}^o + C_{d,2}^o)}{C_d^* - (C_{d,1}^o + C_{d,2}^o)} \right]. \tag{2.4}$$

In other words, collaboration scheme F is efficient if its associated expected cost reduction is similar to the maximum achievable reduction, i.e., $\varepsilon^F = 1$.

2.4.2 Fairness Criteria

A set of axioms has become common in the literature of bargaining problems, where each of them represents some dimension of fairness. Based on the descriptions on Kalai and Smorodinsky (1975), Kalai (1977) and Bertsimas et al. (2011), we give a formal definition of such axioms for the studied problem.

The satisfaction of the following axioms in a planning block can be extrapolated to the whole horizon of the operations pooling. This is a consequence of the independence between the decisions of the different planning blocks. Thus, if a collaboration scheme meets an axiom in a planning block, we say that the operations pooling meets such axiom.

Axiom 2.1. *Individual Rationality. Collaboration scheme F is individual rational if each firm reduces its cost compared with its situation in the absence of the agreement, i.e.*

$$C^F(\mathscr{C}_d, C_d^o) \leq C_d^o. \tag{2.5}$$

This axiom establishes that no firm is hurt by the implementation of the operations pooling.

Axiom 2.2. *Pareto-optimality. Collaboration scheme F is pareto-optimal if no firm can be made better off without making another firm worse off, i.e.* $\nexists C \in \mathscr{C}_d$ *such that* $C \leq C^F(\mathscr{C}_d, C_d^o)$ *and* $C \neq C^F(\mathscr{C}_d, C_d^o)$.

The pareto-optimality imposes that any joint planning decision must avoid unnecessary losses for the firms. Clearly, the planning that minimizes the sum of the costs of the firms meets this criterion, but such planning may be incompatible with other axioms of fairness when transfer payments are not allowed (for example Axiom 2.1). In such case, when a collaboration scheme ensures other axioms of fairness, the pareto-optimality discriminates favourably towards most efficient solutions.

Axiom 2.3. *Symmetry. Let* $T : \mathbf{R}^2 \to \mathbf{R}^2$ *be the permutation operator defined by* $T((a_1, a_2)) = (a_2, a_1)$. *Then,* $T(C^F(\mathscr{C}_d, C_d^o)) = C^F(T(\mathscr{C}_d), T(C_d^o))$.

The symmetry implies that the fair planning decisions cannot be affected by how the firms are named. Consequently, the axiom imposes that firms with identical characteristics should get identical outcomes.

Axiom 2.4. *Invariance with respect to affine transformations of costs. Let* $A : \mathbf{R}^2 \to \mathbf{R}^2$ *be an affine operator defined by* $A(a_1, a_2) = (A_1(a_1), A_2(a_2))$, *with* $A_i(a) = \alpha_i \cdot a + \beta_i$ *and* $\alpha_i \geq 0$. *Then* $A(C^F(\mathscr{C}_d, C_d^o)) = C^F(A(\mathscr{C}_d), A(C_d^o))$.

This axiom establishes that the cost allocation must be invariant to the way that each firm measures the cost of its operations. Nevertheless, there are some criticisms to the pertinence of this axiom in practice. Nydegger and Owen (1974) carried out experiments which show that Axiom 2.4 is frequently violated by subjects. Nydegger and Owen (1974) explain this phenomenon by the fact that *subjects, whenever possible, try to effect an interpersonal comparison of utility*. Thereby, Kalai (1977) proposes a weak version of Axiom 2.4, which is known as homogeneity.

Axiom 2.5. *Homogeneity. Let* $H : \mathbf{R}^2 \to \mathbf{R}^2$ *be the homogeneity operator defined by* $H(h_1, h_2) = (\alpha \cdot h_1, \alpha \cdot h_2)$ *and* $\alpha \geq 0$. *Then* $H(C^F(\mathscr{C}_d, C_d^o)) = C^F(H(\mathscr{C}_d), H(C_d^o))$.

The homogeneity requires that the planning decisions remain identical if the cost of the firms are linearly modified by the same factor. Thereby, the planning decisions of the firms are independent of the unit used to measure the costs. This axiom simplifies Axiom 2.4 (both costs should be modified by the same factor), but it eases the comparison between firms since the measure of the costs is identical for the firms.

Axiom 2.6. *Independence of irrelevant alternatives. If* $\mathscr{C}_d \subseteq \mathscr{C}'_d$ *and* $C^F(\mathscr{C}'_d, C_d^o) \in \mathscr{C}_d$, *then* $C^F(\mathscr{C}_d, C_d^o) = C^F(\mathscr{C}'_d, C_d^o)$.

This axiom states that preferring a planning decision over another one is independent of other available options. Nevertheless, there are objections to this axiom in the literature (see Kalai and Smorodinsky (1975)), which leads to the following alternative axiom.

Axiom 2.7. *Monotonicity. Given the sets $\mathscr{C}_d$ and $\mathscr{C}_d'$ such that $\min\{C_1 \mid (C_1,C_2) \in \mathscr{C}_d\} = \min\{C_1 \mid (C_1,C_2) \in \mathscr{C}_d'\}$, if for any cost associated to the operations of Firm 1 the cost cost that Firm 2 can obtain simultaneously is smaller or equal in $\mathscr{C}_d'$, then $C_2^F(\mathscr{C}_d, C_d^o) \geq C_2^F(\mathscr{C}_d', C_d^o)$.*

The monotonicity establishes that if, for every cost resulting of the operations of Firm 1, the minimum cost that Firm 2 can obtain simultaneously is reduced, then the fair planning decision should imply a reduction of the cost of Firm 2.

The goal of any collaboration scheme should be the satisfaction of the described axioms. Nevertheless, no scheme exists that simultaneously meets all of them (see Nash (1950a) and Kalai and Smorodinsky (1975) for more details). Aiming to cover as many axioms as possible, a vast literature on fairness studies two approaches for implementing collaborative initiatives. The first one is the proportional fairness approach (Nash, 1950a), that leads to a cost allocation in which the percentage decrease in the cost of one firm is larger than the percentage increase in cost of the other firm. This approach satisfies Axioms 2.1 - 2.6. Given the criticisms to Axiom 2.4 and Axiom 2.6 in terms of their pertinence for comparing the relative results between firms, Kalai and Smorodinsky (1975) proposes the Kalai-Smorodinsky solution. The essence of this second approach is derived from the Rawls' theory of justice, in which the agreement prioritizes the firm that is least well off. This approach satisfies Axioms 2.1 - 2.3, 2.5 and 2.7. The Rawlsian theory of justice fits well with the expectations of fairness of partners in business environments. In such environments the success of the partnership is based on building a trusting relationship, where the key is to treat the weaker partner fairly (Kumar, 1996). So, the collaboration schemes that we propose in this research are derived from the Kalai-Smorodinsky solution.

An alternative to the axiomatic characterization of fairness is introduced by Fehr and Schmidt (1999). The authors introduce an utility function to describe the preferences of the partners. This utility of a partner is a parametric function of a firm's own and the imbalance between firms. We adapt the Fehr and Schmidt function to our bargaining problem, so the utility per unit obtained by Firm i is as follows.

$$\begin{aligned} u_i^F(\mathscr{C}_d, C_d^o) = &- c_{d,i}^F(\mathscr{C}_d, C_d^o) - \beta_i \cdot [c_j^F(\mathscr{C}_d, C_d^o) - c_i^F(\mathscr{C}_d, C_d^o)]^+ \\ &- \gamma_i \cdot [c_i^F(\mathscr{C}_d, C_d^o) - c_j^F(\mathscr{C}_d, C_d^o)]^+, \end{aligned} \tag{2.6}$$

where $c_d^F(\mathscr{C}_d, C_d^o) \in \mathbf{R}^2$ denotes the unit costs corresponding to $C_d^F(\mathscr{C}_d, C_d^o)$, i.e., $c_{d,i}^F(\mathscr{C}_d, C_d^o) = C_{d,i}^F(\mathscr{C}_d, C_d^o) / \sum_{t=1}^T d_{i,t}$; $\beta_i \geq 0$ measures the impact of a disadvantageous inequality on the utility of Firm i; $\gamma_i \geq 0$ measure the loss from advantageous inequality. So, β_i and γ_i measures the inequality aversion of a firm. The values of these measures are constrained by two conditions. First, $\beta_i \geq \gamma_i$, i.e. negative deviations from the reference outcome count more than positive deviations. Second, $\gamma_i \leq 1$, on the contrary Firm i is prepared to throw away one dollar in order to reduce his advantage relative to Firm j which seems very implausible.

In our problem, the demand is unknown when firms agree on the collaboration scheme, so we propose the following indicator to compare the impact of the fairness and the efficiency:

$$U_i^F = E_D\left[u_i^F(\mathscr{C}_d, C_d^o)\cdot\sum_{t=1}^{T} d_{i,t}\right]. \tag{2.7}$$

Clearly, implementing operations pooling based on the Fehr and Schmidt function may be difficult because such function is highly dependent on the capability of finding the exact values of β_i and γ_i. However, we can utilize indicator U_i^F for evaluating the relative performance of collaboration schemes under different levels of inequality perceptions. So, we can determine whether the operations pooling should be based on one scheme or another according to the values of β_i and γ_i.

2.4.3 Risk Measures

When future partners aim to address the risk of operations pooling, the first step is to agree on the risk they would like to control. Since the demand in a planning block is uncertain, the firms faces two types of risk: the market risk, which is inherent to the demand; and the operational risk, which is related to the per unit cost resulting of the production activities carried out to satisfy the demand. Since the firms cannot control their market risk in the short term, the risk of operational pooling should isolate the operational risk from the market risk. Thus, we propose that firms focus on their average unit costs when measuring the risk of operations pooling.

The firms also have to reach a consensus on how to measure their risks. Artzner et al. (1999) propose a set of axioms that a risk measure might have. If a risk measure satisfies such axioms, then it is called a coherent risk measure, this concept has become common for problems of portfolio optimization. We adjust those axioms to operations pooling. To this end, we say that the function $\rho(F) \in \mathbf{R}^2$ measures the risk of implementing collaboration scheme F, where $\rho_i(F)$ is the risk of Firm i.

Axiom 2.8. *Monotoniciy of risk. Given two collaboration schemes F and F', if $c_{d,i}^F(\mathscr{C}_d, C_d^o) \le c_{d,i}^{F'}(\mathscr{C}_d, C_d^o)$ for any realization of the demand d, then $\rho_i(F) \le \rho_i(F')$.*

This axiom establishes that if the unit cost of Firm i under scheme F is smaller than under scheme F' for all the possible scenarios, then the risk of Firm i under scheme F is smaller than under scheme F'.

Axiom 2.9. *Positive homogeneity. Given two collaboration schemes F and F', if $c_i^{F'}(\mathscr{C}_d, C_d^o) = k\cdot c_i^F(\mathscr{C}_d, C_d^o)$ for any realization of demand d, then $\rho_i(F') = k\cdot\rho_i(F), \forall k > 0$.*

The positive homogeneity states that if the unit cost of Firm i resulting from scheme F is doubled by scheme F', then the risk of Firm i derived from implementing scheme F' is the double of the risk when implementing scheme F.

Axiom 2.10. *Translation invariance. Given two collaboration schemes F and F', if $c_i^{F'}(\mathscr{C}_d, C_d^o) = k + c_i^F(\mathscr{C}_d, C_d^o)$ for any realization d, then $\rho_i(F') = k + \rho_i(F)$.*

In general, the risk measure represents the amount of money that a firm has to add to its position in order to make it acceptable.

Axiom 2.11. *Risk subadditivity. Given three collaboration schemes F, F' and F'', if the scheme F'' consists of utilizing F with a probability p and F' with a probability $1-p$, then $\rho_i(F'') \leq p \cdot \rho_i(F) + (1-p) \cdot \rho_i(F'), \forall i$.*

This axiom establishes that the risk of combining two schemes cannot get any worse than the weighted sum of the two risks separately. This axiom is coherent with the diversification principle commonly used in portfolio problems, however, such principle has a limited compatibility with collaboration problems.

A widely used measure of risk in portfolio problems is the value at risk. This measure computes a threshold value for a given probability level α, such that the unit cost of a firm exceeds this value with a probability α. In the context of operations pooling, we define the value at risk implementing collaboration scheme F as:

$$VaR_i^F(\alpha) = \inf\left\{c \in \mathbb{R} \mid P\left[c_{d,i}^F(\mathscr{C}_d, C_d^o) \leq c\right] \geq \alpha\right\} \tag{2.8}$$

The value at risk satisfies Axioms 2.8 - 2.10, but it may not respect Axiom 2.11, therefore, this indicator is not a coherent measure of risk. Another limitation of the value at risk is that omits the impact of the costs exceeding the threshold value c. In other words, the tail end of the distribution of costs is not assessed by this indicator.

Aiming to overcome the shortcomings of the value at risk, Rockafellar and Uryasev (2000) introduce the conditional value at risk. Although the formulation of this measure is similar to the value at risk, the conditional value at risk computes the expected cost of the scenarios in the tail of the percentile. Moreover, this measure of risk also satisfies the Axiom 2.11, therefore, it is a coherent risk measure. In the context of operations pooling, we define the conditional when implementing scheme F as:

$$CVaR_i^F(\alpha) = E\left[c_{d,i}^F(\mathscr{C}_d, C_d^o) \mid c_{d,i}^F(\mathscr{C}_d, C_d^o) \geq VaR_i^F(\alpha)\right] \tag{2.9}$$

Both indicators, value at risk and conditional value at risk, may be useful for measuring the risk in operations pooling. In particular, firms can utilize these measures to control the fluctuations of the cost resulting from pooling operations.

2.5 Collaboration Schemes

As we discussed in section 2.2, launching agreements thqt do not require transfer payments is in general easier for the firms. But the absence of payments makes it more complicated to combine efficiency and fairness. Indeed, efficient solutions may violate some fairness criteria, in particular, individual rationality. Therefore, the firms are searching for a collaboration scheme that meets fairness criteria, while avoiding to loose too much efficiency.

In this section we propose two collaboration schemes, Max-min Normalized Savings Scheme (*NS*) and Min-max Unit Cost Scheme (*UC*). Both schemes are based on the Rawlsian notion of fairness, i.e. their fundamental premise is to accept inequalities on the allocation of costs only if such inequalities benefit the worst-off member in society. Thus, Scheme *NS* and Scheme *UC* aim to improve the partner that is least well off.

2.5.1 Individual Rationality Scheme

This scheme assumes that the individual rationality is a necessary condition for implementing a collaborative initiative between partners. The partners minimize the global cost subject to constraint 1.5. Thus, $C_d^{IR}(\mathscr{C}_d, C_d^o)$ corresponds to the solution of the following model.

$$\min \left\{ C_1 + C_2 \mid C_i(\mathscr{C}_d, C_d^o) \leq C_{d,i}^o \forall i = 1,2 \wedge (C_1, C_2) \in \mathscr{C}(d) \right\}. \tag{2.10}$$

Even though this scheme is weak in terms of satisfying the discussed fairness criteria, it is a good benchmark for other schemes, because it satisfies the minimum conditions that makes the collaboration attractive for the firms (the collaboration does not hurt partners).

2.5.2 Max-min Normalized Savings Scheme

In order to utilize an scale that is common for both firms, we measure the proportional reduction of costs by the ratio between two values: (i) the reduction of costs with respect to working in isolation, i.e. $(C_{d,i}^o - C_{d,i}^{NS}(\mathscr{C}_d, C_d^o))$ and (ii) the largest potential reduction that a firm can achieve by implementing the operations pooling, i.e. $(C_{d,i}^o - \bar{C}_{d,i})$, where $\bar{C}_{d,i} = \min\{C_1 \mid C_i \leq C_{d,i}^*, \forall i = 1,2 \wedge (C_1, C_2) \in \mathscr{C}_d\}$ (the minimum cost that Firm *i* can achieve by implementing pooling while Individual Rationality Axiom is satisfied). As we are in the line of the Rawlsian notion of fairness, the objective of Scheme *NS* is to maximize the minimum ratio of cost reduction of the firms. Thus, $C_d^{NS}(\mathscr{C}_d, C_d^o)$ corresponds to the solution of the following model.

$$\max \left\{ \phi \mid \phi \leq \frac{C^o_{d,i} - C_i}{C^o_{d,i} - \bar{C}_{d,i}}, \forall i = 1,2 \wedge (C_1, C_2) \in \mathscr{C}(d) \right\}. \tag{2.11}$$

Given that $\mathscr{C}_d$ is the mapping of the costs allocations resulting from $\mathscr{X}_d$, we can obtain the solution of problem (2.11) by solving 6 MILP problems: first, we solve the two single *CLSP* linked to $C^o_{d,1}$ and $C^o_{d,2}$; second, we solve the two MILP related to $\bar{C}_{d,1}$ and $\bar{C}_{d,2}$; third, we solve the MILP formulation associated to problem (2.11); without loss of generality, let us assume that Firm 1 obtained the minimum ratio; finally, as firms aim to avoid potential inefficiencies, we solve a MILP that minimizes we the cost of Firm 2, but assuming that the planning decisions for Firm 1 are fixed.

2.5.3 Min-max Unit Cost Scheme

In this scheme we utilize the average unit cost to measure the performance of a firm. Utilizing the unit cost as a performance measure is consistent with the discussion about the risk of pooling operations (see Section 2.3), decision makers can isolate as much as possible the risk of the operations pooling from factors that they cannot control, such as demand uncertainty.

As our purpose is to follow the Rawlsian notion of fairness, implementing Scheme *UC* leads to minimizing the maximum unit average cost among firms, while satisfying the individual rationality conditions. Thus, through applying this scheme, the operations pooling focuses on helping the firm facing the more critical situation in terms of its average unit cost. Thus, $C^{UC}_d(\mathscr{C}_d, C^o_d)$ corresponds to the solution of the following model.

$$\min \left\{ \psi \mid \psi \geq \frac{C_i}{\sum_{t=1}^{T} d_{i,t}}, \forall i = 1,2 \wedge C_i \leq C^*_{d,i}, \forall i = 1,2 \wedge (C_1, C_2) \in \mathscr{C}(d) \right\}. \tag{2.12}$$

As for Scheme *NS*, we can obtain the joint production plan derived from Scheme *UC* by solving several MILP problems: first we solve the two MILP problem associated to $C^o_{d,1}$ and $C^o_{d,2}$; second we avoid inefficiencies by solving lexicographically problem (2.12).

When setting up operations pooling based on Scheme *UC*, some difficulties may arise if there exists a significant difference between the magnitude of cost of the firms. However, when firms are aware in advance of natural differences between their unit average costs, the proposed formulation can be easily adjusted to include such differences. Thus, the properties in terms of fairness of the proposed scheme will not be affected. For instance, if the firms expect that the unit cost of Firm 1 is systematically k_c times higher than the unit cost of Firm 2, then we replace $\psi \geq \frac{C_2}{\sum_{t=1}^{T} d_{i,t}}$ by

$\psi \geq k_c \cdot \frac{C_2}{\sum_{t=1}^{T} d_{i,t}}$ in problem 2.12.

2.5.4 Efficiency, Fairness and Risk of Collaboration Schemes

Given that the operations resulting of implementing Scheme NS and Scheme UC are always feasible solutions for model (2.10), we can easily conclude that the efficiency of Scheme IR is higher than for Scheme NS and Scheme UC, i.e. $\varepsilon^{IR} \geq \varepsilon^{NS}$ and $\varepsilon^{IR} \geq \varepsilon^{UC}$.

Although implementing Scheme IR is more attractive for the firms in terms of the efficiency of the operations pooling, Scheme UC is efficient in reducing the inequalities that may arise as a result of pooling operations. In order to explain the advantages of Scheme UC, we measure the attractiveness of a scheme by the utility function introduced in equation (2.7).

Theorem 2.1. *Let F be a scheme that satisfies the individual rationality and the symmetry axioms. Also, let Firm 1 and Firm 2 be identical firms, i.e., their operations are characterized by identical parameters and their demands are identically distributed. Thus,*

1. *for any inequality aversion of the firms, if Scheme F is less efficient than Scheme UC, then the utility level that the firms achieve by implementing Scheme F is lower than for Scheme UC, in other terms,* $\varepsilon^F < \varepsilon^{UC} \implies U_i^F < U_i^{UC}$;
2. *there exists an equality aversion level* $\beta_i^{F,UC}$ *such that for any inequality aversions lower than* $\beta_i^{F,UC}$, *if Scheme F is more efficient than Scheme UC, then the utility level that the firms achieve by implementing Scheme F is higher than for Scheme UC; also, for inequality aversions greater than or equal to* $\beta_i^{F,UC}$, *if Scheme F is less efficient than Scheme UC, then the utility level that the firms achieve by implementing Scheme F is lower than for Scheme UC.*

Proof. We start by expressing U_i^F as the sum of three terms: $U_i^F = -E_D[C_{d,i}^F(\mathscr{C}_d, C_d^o)] - \beta_i \cdot U_{i,1}^F - \gamma_i \cdot U_{i,2}^F$, where $U_{i,1}^F = E_D\left[[c_j^F(\mathscr{C}_d, C_d^o) - c_i^F(\mathscr{C}_d, C_d^o)]^+ \cdot \sum_{t=1}^T D_i\right]$ and $U_{i,2}^F = E_D\left[[c_i^F(\mathscr{C}_d, C_d^o) - c_j^F(\mathscr{C}_d, C_d^o)]^+ \cdot \sum_{t=1}^T D_i\right]$.

Our interest is to find the conditions of β_i such that $U_i^{UC} - U_i^{F'} \geq 0$, i.e., $\beta_i \geq \frac{E_D[C_{d,i}^{UC}(\mathscr{C}_d, C_d^o)] - E_D[C_{d,i}^{F'}(\mathscr{C}_d, C_d^o)]}{U_{i,1}^{F'} - U_{i,1}^{UC} + k_i \cdot (U_{i,2}^{F'} - U_{i,2}^{UC})}$, where we assume that $\gamma_i = k_i \cdot \beta_i$.

Given that the solution of Scheme UC corresponds to the individual rational and pareto-optimal operations that minimizes the difference between the average cost of the firms, $U_{i,1}^{F'} - U_{i,1}^{UC} \geq 0$ and $U_{i,2}^{F'} - U_{i,2}^{UC} \geq 0$, so $U_{i,1}^{F'} - U_{i,1}^{UC} + k_i \cdot (U_{i,2}^{F'} - U_{i,2}^{UC}) \geq 0$.

On the other hand, if Scheme UC is more efficient than F, then $E_D[C_{d,i}^{UC}(\mathscr{C}_d, C_d^o)] - E_D[C_{d,i}^{F'}(\mathscr{C}_d, C_d^o)] < 0$, so any $\beta_i \geq 0$ satisfies $U_i^{UC} - U_i^{F'} \geq 0$. If Scheme UC is not more efficient than F, $E_D[C^{UC}_{d,i}(\mathscr{C}_d, C^o_d)] - E_D[C^{F'}_{d,i}(\mathscr{C}_d, C^o_d)] \geq 0$, so $(E_D[C^{UC}_{d,i}(\mathscr{C}_d, C^o_d)] - E_D[C^{F'}_{d,i}(\mathscr{C}_d, C^o_d)])/(U^{F'}_{i,1} - U^{UC}_{i,1} + k_i \cdot (U^{F'}_{i,2} - U^{UC}_{i,2})) \geq 0$. □

Theorem 2.1 leads to interesting conclusions about how efficiency and fairness determine the attractiveness of Scheme *IR*, Scheme *NS* and Scheme *UC*. From the first part of Theorem 2.1, we conclude that firms should pool operations based on Scheme *UC* instead of Scheme *NS* when the efficiency of this last scheme is lower than for Scheme *UC*. The second part of the theorem leads to conclude that Scheme *UC* is the most attractive when the firms have high inequality aversions. Moreover, the relative attractiveness of Scheme *UC* and Scheme *IR* is characterized for a threshold value in terms of the inequality aversion. This behavior can be extrapolated to the comparison between Scheme *NS* and Scheme *IR*, when the efficiency of this last scheme is greater than for Scheme *UC*.

Further, the firms can get advantages from implementing Scheme *IR*, Scheme *NS* or Scheme *UC* in terms of their risk. This idea is summarized by Theorem 2.2.

Theorem 2.2. *If Firm 1 and Firm 2 are identical firms and Scheme F satisfies individual rationality and symmetry axioms, then:*

1. *the risk resulting from pooling operations is equally allocated between firms,*
2. *the risk resulting from pooling operations is lower than or equal to the risk of working in isolation.*

Proof. Given that the firms are identical and Scheme F satisfies the Axiom of Symmetry, the implementation of such scheme implies that the unit cost of the firms are identically distributed. Therefore, $VaR^F_1(\alpha) = VaR^F_2(\alpha)$, and $CVaR^F_1(\alpha) = CVaR^F_2(\alpha)$.

Further, since Scheme F satisfies the Axiom of Individual Rationality, the implementation of such scheme implies that the unit costs of both firms are decreased for any demand scenario. Consequently, $VaR^F_i(\alpha) \leq VaR^o_i(\alpha)$ and $CVaR^F_i(\alpha) \leq CVaR^o_i(\alpha)$, $\forall i = 1, 2$. □

In addition to the results of Theorem 2.2, in Section 2.6 we show through numerical experiments that Scheme *UC* is in general more efficient in reducing the risk of the firms than other schemes. This is a result of the reduction on the dispersion of the results of each firm when implementing Scheme *UC*, because its objective function is based on the min-max of the average cost of the firms.

2.6 Numerical Experiments

With the aim of studying the effects of pooling operations, we develop numerical experiments that provide evidence of the advantages and disadvantages of the collaboration schemes introduced in Section 2.5. Specifically, we analyze the efficiency, the fairness and the risk associated with those schemes. Further, our numerical experiments help to glimpse the characteristics of the firms which are potentially more advantageous when pooling operations.

We represent different characteristics of the partners by setting instances of their production parameters. We use the following assumptions for building such instances:

- The firms are identical, i.e., their cost parameters, production capacities and probability distribution of their demands are identical.
- The cost parameters and the production capacity of a firm do not vary with periods, i.e. $f_{i,t} = f, p_{i,t} = p, h_{i,t} = h, b_{i,t} = b, CAP_{i,t} = CAP, \forall i,t$.
- The demand that a firm faces in a period is i.i.d. such that $D_{i,t} \sim \Gamma(\frac{\bar{d}}{CV^2}, \cdot CV^2)$, in which $\bar{d}$ and CV are the average value and the coefficient of variation of the demand in a period, respectively.
- We assume that the firms face three demand loads: low, medium and high. We derive such levels from the ratio between the expected demand and the production capacity per period, what we call capacity tightness (CT). Therefore, $CT = \frac{\bar{d}}{CAP}$ and we set the three levels of demand load to $CT = 0.2, 0.5, 0.8$.

In order to explore different scenarios of operations pooling, we vary the parameters of the firms. Each scenario is constructed from a Base set of parameters (see the second column of the Table 2.1), and we modify each parameter according to the third column of Table 2.1. For each scenario, we generate randomly 2,000 demand instances in such way we obtain reliable values for the studied measures of efficiency, fairness and risk.

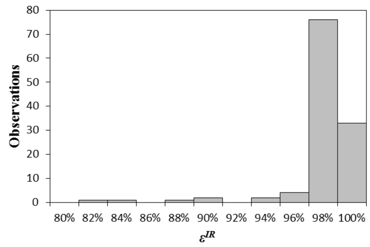
Parameter	Base Set	Range of Values
T	10	$\{1, 2, ..., 10\}$
f	100	100
CAP	100	100
p	$p = f \cdot k_p$, $k_p = 0.1$	$k_p = \{0.1, 0.2, ..., 1.0\}$
h	$h = p \cdot k_h$, $k_h = 1.5$	$k_h = \{0.5, 1.0, ..., 5.0\}$
b	$b = 2 \cdot h$	$b = 2 \cdot h$
CT	$\{0.2, 0.5, 0.8\}$	$\{0.2, 0.5, 0.8\}$
CV	0.4	$\{0.2, 0.4, ..., 2.0\}$

Table 2.1: *Parameters of the Problem.*

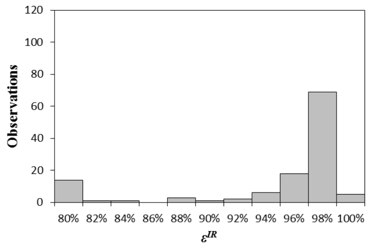
In terms of the cost structure of the firms, we denominate $CS1$ the cost structure in which the unit production cost is supported by the firm for whom the product is made and $CS2$ the cost structure in which the unit production cost is linked to the production lines.

2.6.1 Efficiency.

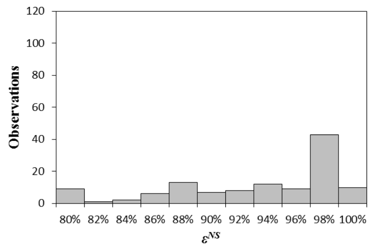
We know that Scheme *IR* is more efficient than Scheme *NS* and Scheme *UC*, but we cannot conclude in advance about the relative efficiency between schemes. Our simulations provide evidence about such relative efficiency and show the impact of the cost structure on the efficiency of the operations pooling. Figure 2.1 summarizes the observed efficiency for the instances considered.



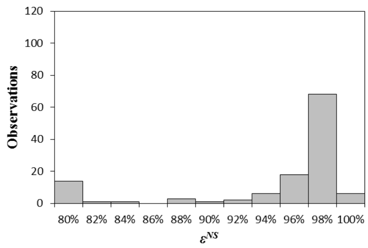
(a) Scheme *IR*. Cost Structure *CS*1.



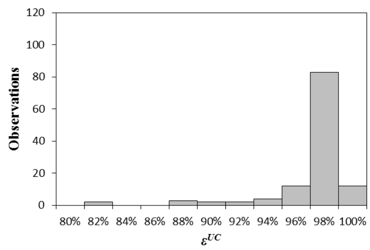
(b) Scheme *IR*. Cost Structure *CS*2.



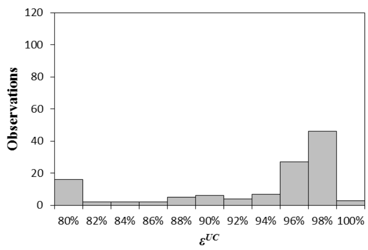
(c) Scheme *NS*. Cost Structure *CS*1.



(d) Scheme *NS*. Cost Structure *CS*2.



(e) Scheme *UC*. Cost Structure *CS*1.



(f) Scheme *UC*. Cost Structure *CS*2.

Figure 2.1: *Observed Cost Efficiency of Different Collaboration Schemes and Cost Structures.*

The main insights we can derive from Figure 2.1 are the following:

- As we could expected, Scheme *NS* and Scheme *UC* are less efficient than Scheme *IR*, but their efficiency is still high. The efficiency of Scheme *UC* is greater than 90% in most of the observed instances under the cost structure $CS1$. The efficiency of this scheme decreases with the cost structure $CS2$, but the efficiency is below 80% in less than 1/6 of the observations. The efficiency of Scheme NS goes below 80% in a few scenarios. Moreover, under the cost structure $CS2$, the efficiency of Scheme NS is similar to the efficiency of IR.

- Scheme UC is more efficient than Scheme NS under the cost structure $CS1$. The inverse relation occurs under the cost structure $CS2$. Under the cost structure $CS1$, $\varepsilon^{UC} \geq \varepsilon^{NS}$ in 89.1% of the simulated scenarios. $\varepsilon^{NS} \geq \varepsilon^{UC}$ in all the scenarios under $CS2$.

- The collaboration schemes are less efficient under the cost structure $CS2$. This loss of efficiency is born of the difficulties of achieving a balanced allocation of cost when the firms operate under such cost structure. Indeed, under cost structure $CS1$ the firms can allocate their unit production costs to minimize their inequalities without affecting the efficiency, however, such reallocation is not possible under the cost structure $CS2$.

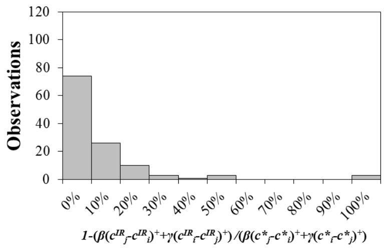
2.6.2 Fairness

We use the Fehr and Schmidt utility function to study the inequality arising from implementing a collaboration scheme. In particular, we measure the observed normalized gap between the Fer and Smith inequality derived from a collaboration scheme and the Fer and smith inequality when minimizing the joint costs. In other words, we keep track of $E_D\left[1-\frac{\beta_i\cdot[c_j^F-c_i^F]^+ +\gamma_i\cdot[c_i^F-c_j^F]^+}{\beta_i\cdot[c_j^*-c_i^*]^+ +\gamma_i\cdot[c_i^*-c_j^*]^+}\right]$. Larger values of this indicator implies that the inequality reduction achieved by implementing Scheme F is has a high impact on the utility of the firms. Figure 2.2 shows the value of this indicator for different level of inequality aversion. Note that, we have set $\beta_i = 0.1$ and $\frac{\gamma_i}{\beta_i} = 0.5$.

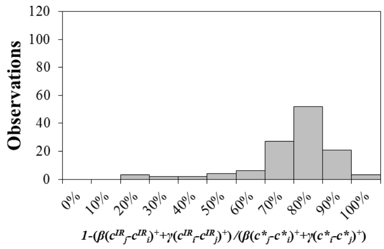
We obtain the following insights from Figure 2.2:

- Scheme UC becomes more convenient than Scheme IR as the inequality aversion of the firms grows. This emphasizes the result of Theorem 1, i.e. when the fairness concerns are high, firms should plan their operations pooling by utilizing Scheme UC.

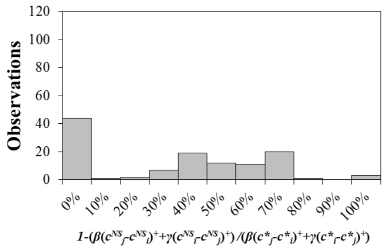
- Scheme NS is dominated by Scheme NS and Scheme IR under the cost structures $CS1$ and $CS2$, respectively. On the one hand, the results related to efficiency and inequality aversion shows that $U^{NS} \leq U^{UC}$ for cost structure $CS1$. On the other hand, the efficiency and the fairness derived from Scheme IR and Scheme UC are almost the same for the cost structure $CS2$.



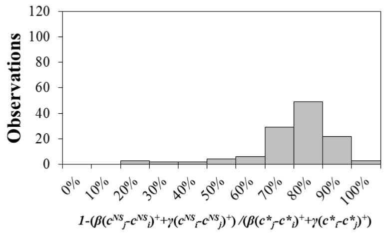
(a) Scheme *IR*. Cost Structure *CS*1.



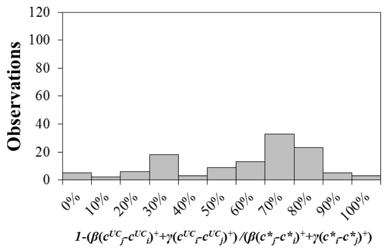
(b) Scheme *IR*. Cost Structure *CS*2.



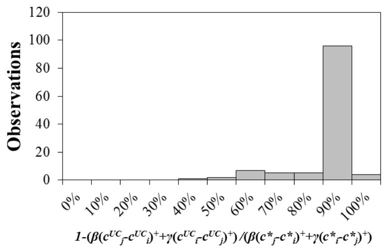
(c) Scheme *NS*. Cost Structure *CS*1.



(d) Scheme *NS*. Cost Structure *CS*2.



(e) Scheme *UC*. Cost Structure *CS*1.



(f) Scheme *UC*. Cost Structure *CS*2.

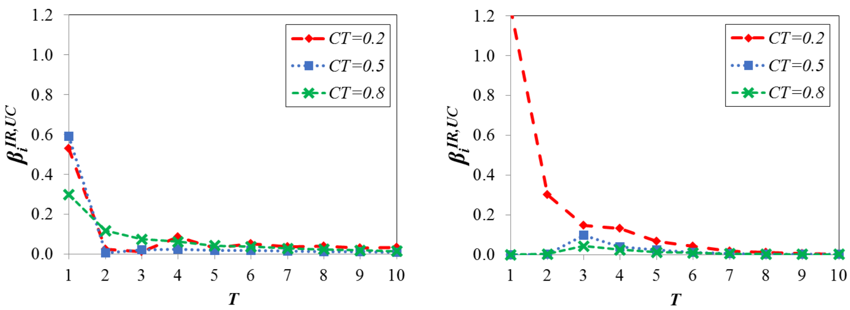
Figure 2.2: *Observed Fehr and Schnidt Utility for Different Inequality Aversions, Collaboration Schemes and Cost Structures.*

- The impact of fairness is stronger under the cost structure *CS*2. This phenomenon occurs because the low divisibility of the costs when firms pool operations under the cost structure *CS*2. Thus, the globally optimal solution may result in unbalanced allocation of costs. This unbalance is reduced by implementing collaboration Scheme *IR*, and in particular, by implementing Scheme *UC*.

Further, we are interest to study the characteristics of the operations pooling that make the implementation of Scheme *UC* worthwhile for the firms. With this in mind, we track the observed value of $\beta_i^{IR,UC}$ in our simulations. Figures 2.3 - 2.6 summarize the relation between the characteristics of the firms and $\beta_i^{IR,UC}$. Here, we provide an analysis for each parameter of the firms.

Planning block length. Figure 2.3 shows that the efficiency of Scheme UC is significantly reduced when the operations pooling is planned for a short planning block. In such scenarios, the Scheme UC will be attractive for the firms only if the inequality aversion of these firms is high.

In particular, since the efficiency is significantly reduced under the cost structure $CS2$, the inequality reductions may not be sufficient attractive to compensate the loss of efficiency. Also under the cost structure $CS2$, $\beta_i^{IR,UC} \sim 0$ for $T = 1,2$. This phenomenon occurs because the operations derived of implementing Scheme IR and Scheme UC are similar, and therefore, Scheme UC does not bring additional benefits from balancing the cost allocation.



(a) Cost Structure $CS1$.

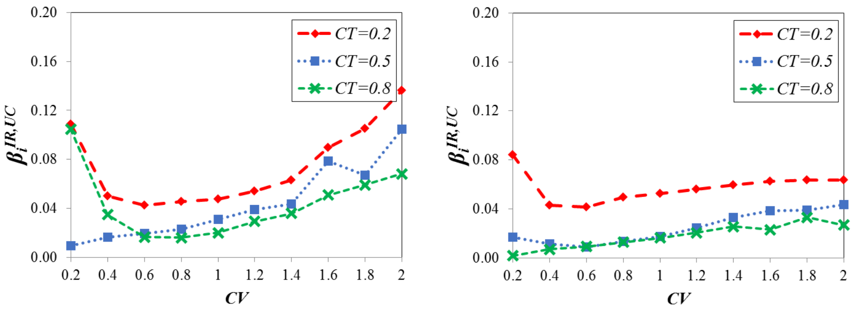
(b) Cost Structure $CS2$.

Figure 2.3: *Observed Values of $\beta_i^{IR,UC}$ For Different Values of T.*

Coefficient of variation of the demand. As we can observe in Figure 1.4(a), the attractiveness of Scheme UC decreases when the variability of the demand is high or low. In such scenarios the Scheme UC will be attractive for the firms only if the inequality aversion of these firms is high. This is caused by two facts: on the one hand, the efficiency of Scheme UC is decreasing with the CV; on the other hand, implementing Scheme IR results in more unbalanced cost allocations when the CV is high.

The effect of the demand variability on the impact of the inequality aversion on the relative attractiveness between Scheme IR and Scheme UC is less significant under the cost structure CS_2. This is because implementing Scheme IR under such cost structure leads to highly unbalanced cost allocations.

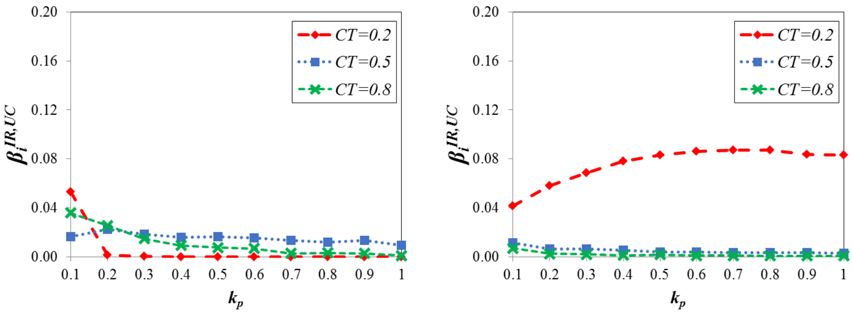
Production cost. As we observe in Figures 2.5 - 2.6 In general the convenience of Scheme UC compared to Scheme IR is stable to changes on the costs of the firms. $\beta_i^{IR,UC}$ becomes higher for low values of k_p and k_h, because in such scenarios the



(a) Cost Structure $CS1$. (b) Cost Structure $CS2$.

Figure 2.4: *Observed Values of* $\beta_i^{IR,UC}$ *For Different Values of* CV.

fixed production costs are relatively higher than other production costs, and therefore, a balance cost allocation is less likely.



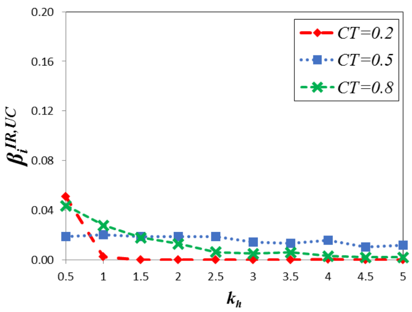
(a) Cost Structure $CS1$. (b) Cost Structure $CS2$.

Figure 2.5: *Observed Values of* $\beta_i^{IR,UC}$ *For Different Values of* k_p.

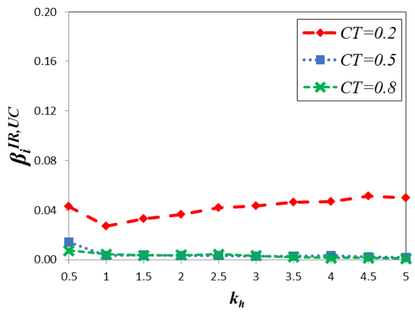
Capacity tightness. We can clearly observe in Figures 2.3 - 2.6 that the value of $\beta_i^{IR,UC}$ is higher when the capacity tightness is low. This relation is explained by the fact that the efficiency gap between Scheme *IR* and Scheme *UC* is higher when the demand is low, since the production is more sporadic and the cost reductions from operations pooling are more related to the fixed production costs. Thus, balancing the allocation of costs by implementing Scheme *UC* may be not attractive for the firms, at least the fairness concerns become higher.

2.6.3 Risk of the collaboration.

We use the conditional value at risk to study the risk of operations pooling based on different collaborations schemes. In particular, we measure the proportion of the



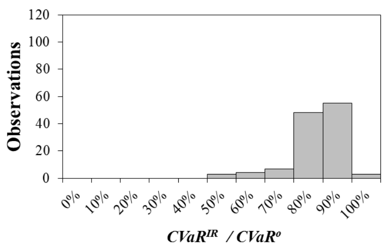
(a) Cost Structure $CS1$.



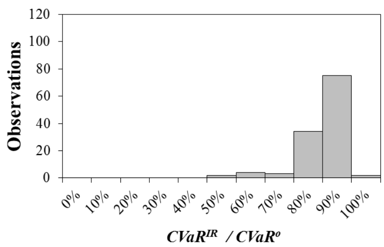
(b) Cost Structure $CS2$.

Figure 2.6: *Observed Values of* $\beta_i^{IR,UC}$ *For Different Values of* k_h.

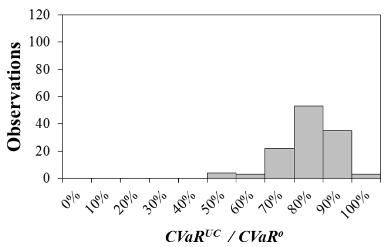
risk achieved by a collaboration scheme ($CVaR_i^F(\alpha)$) with respect to the the risk of working in isolation ($CVaR_i^o(\alpha)$) for $\alpha = 95\%$. In other words, we keep track of $\frac{CVaR_i^F(\alpha)}{CVaR_i^o(\alpha)}$. Figure 2.7 shows the value of this indicator indicator $\frac{CVaR_i^F(\alpha)}{CVaR_i^o(\alpha)}$. Note that, we exclude Scheme *NS* from the analysis, because such scheme is clearly dominated by Scheme *IR* and Scheme *UC* in terms of efficiency and fairness.



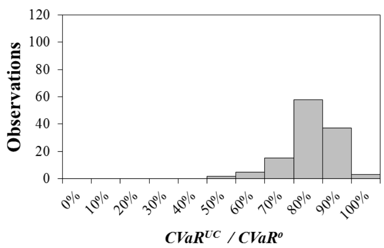
(a) Scheme IR. Cost Structure $CS1$.



(b) Scheme IR. Cost Structure $CS2$.



(c) Scheme UC. Cost Structure $CS1$.



(d) Scheme UC. Cost Structure $CS2$.

Figure 2.7: *Observed Risk for Risk Level=95%, Collaboration Schemes and Cost Structures.*

The main conclusions from Figure 2.7 are as follows:

- As we expect from Theorem 2.2, the firms reduce their risk by pooling operations. We obtain that $CVaR_i^{IR}(\alpha) < CVaR_i^{o}(\alpha)$ and $CVaR_i^{UC}(\alpha) < CVaR_i^{o}(\alpha)$ for all the analyzed instances.

- In general, the operations derived from implementing Scheme UC are less risky for the firms in comparison to Scheme IR. Under the cost structure $CS1$, the condition $CVaR_i^{UC}(\alpha) \leq CVaR_i^{IR}(\alpha)$ is satisfied for all the scenarios. Under the cost structure only 3 instances provide higher risk by implementing Scheme UC, but the gap with respect to Scheme IR is not significant.

2.7 Summary and Conclusions

In this Chapter we studied the collaboration initiatives in which firms pool their operations. The firms should reach a consensus on the scheme by which they will jointly plan operations. this scheme must ensure that the joint gains of the firms will surpass the effort of implementing it. However, the sustainability of the collaboration depends also on that the gains are fairly allocated between firms.

Traditional approaches for collaboration in operations assume that partners can make transfer payments to balance the allocation of gains. We propose collaboration schemes which are flexible in terms of the existence of such payments. Thus, a trade-off between efficiency and fairness arises, and consequently, there is a direct link between the operations management and the collaboration scheme agreed by the firms.

We propose a collaboration scheme that determines the joint operations by minimizing the maximum unit production costs of the partners. We show that even though this scheme may may reduce the efficiency of the collaboration in comparison to the global optimization, it is successful in balancing such efficiency and the fairness of the cost allocation.

Moreover, when the collaboration is carried out under uncertainty, the risk derived from pooling operation is less than the risk faced by the firms by working in isolation. In particular, our scheme outperforms other schemes in terms of the risk for the firms. So, this scheme become more interesting for the firms.

Implementing schemes that take into account the flexibility in transfer payments, the fairness in the allocation of gains and the effects on the risk of the firms, help to decision makers to bring to practice theoretical models of collaboration. In other words, decision makers can utilize our methodology to keep successful partnerships.

We suggest three main branches as future researches derived from our work:

- We consider interesting to extend our work to collaboration in fully-decentralized systems under asymmetric information. Our work assume that the firms share the full information about their demands each time they jointly plan operations. This scenario may corresponds to environments when there is a central decision maker or when the firms can collate the demand of each other in the short term. In other environments the firms may reveal demand information different to the actual amounts, because they expect to benefit from such distortion. Solving this type of problem will requires to combine Game Theory and Bilevel Programming (first level is related to the decision of revealing demand and in the second level the firms plan their joint operations based on the reveal demands).

- Extending our methodology to operations pooling between three or more firms could be interesting from theoretical and practical point of view. The essential notion of fairness behind our proposed scheme will not vary for that problem, but elements related to the stability of the agreement should be included in our models, i.e. the gains obtained in the grand coalition between firms have to surpass the gains of each sub-coalition. this additional dimension of the problem will increase the computational time of jointly planning operations.

- Our collaboration scheme could be applied in other settings in which firms can benefit from the economies of scale resulting from a collaboration. For example, it could be interesting to study the potential collaboration in a distribution network among a set of firms, so that such firms can gain from reducing the cost associated to their fleet size.

Chapter 3

Pricing and Operations Planning Under Competition

3.1 Introduction

Offering the right price to the customer is one of the most challenging tasks for a firm. With this in mind, more firms in the retail or other industries adopt dynamic pricing strategies. Elmaghraby and Keskinocak (2003) explain this phenomenon by three facts: (i) an increased availability of demand data, (ii) ease of changing prices due to new technologies, and (iii) an availability of decision-support tools for analyzing demand data and for dynamic pricing.

When making pricing decisions, a firm needs a strong coordination between its marketing and operations department. On the one hand, the marketing department should measure how the customers value the products offered by the firm, and in particular, the influence of the price on their demands. On the other hand, the operations department organizes the production activities such that the firm satisfies the demand at minimum cost. If the coordination is weak, the firm will carry out operations that may not provide the quantities requested by the customers, or the marketing department may set prices that lead to revenues which may not be enough large to cover the production costs.

Furthermore, even when a firm is able to coordinate its internal decisions, its performance also depends on the interaction with other agents belonging to its own supply chain or belonging to different supply chains. Making decisions without taking such interactions into account may lead to sub-optimal results. In particular, pricing decisions are highly dependent on the interaction between agents belonging to different supply chains, because the items produced by these agents can have a substitution effect on each other. This is a classical case of competition, where the pricing decisions of a group of competitors will arise from the equilibrium between them.

In this work, we study the problem of two competitors making pricing and operations decisions for similar items. The main assumptions of our work are as follows.

First, we model the operations of the firms as a dynamic lot sizing problem (LSP). The LSP is a well known production-inventory problem that establishes when and how much to produce in order to minimize set-up, production and holding costs for a determined number of discrete periods (see Wolsey (1995), Karimi et al. (2003), Brahimi et al. (2006) for an extended literature review of this problem). The presence of setup costs leads to economies of scale in the production costs, which makes the competition more intense (Cachon and Harker, 2002). The reason for the strong competition is explained by two facts: (i) the firms must capture a positive threshold of demand or else it is not profitable, and (ii) the price competition increases, since reducing prices may lead to a demand increase, which results in a reduction of the average cost per unit. Furthermore, the assumption of a LSP for modeling operations has implications in terms of the complexity for getting equilibria. Given that the LSP is a discrete problem, determining prices and operations may be a hard task even in the monopolistic case. Consequently, the computation of an equilibrium between firms could be even harder.

Second, we assume there is a substitution effect between items produced by the firms, however, the items are not necessarily perfect substitutes. We model the demand as a linear function of the prices of both the item itself and the alternative item. The magnitude of the parameter associated to the opponent's price determines the degree of substitution between the items.

Our third assumption is that each of the firms maximizes its profits over a discrete set of prices. Such price simplification responds to the fact that the success of dynamic pricing depends on how the consumer understands and manages pricing signals (Bonsall et al., 2007). Also, a reduced set of prices helps in terms of the pricing transparency, promoting customer satisfaction and its loyalty in the long-term (Mauri, 2007). Lanquepin-Chesnais et al. (2012) describe two of the most common examples of discrete set of prices: discounts in fashion business are often 70%, 50% or 20%; and the low-demand, medium-demand and high-demand prices in the power supply industry.

The assumptions to model operations as a LSP and discrete prices have a direct impact on how to achieve a price equilibrium between firms. Such assumptions imply that the pay-off of a firm cannot be expressed as a close function of the opponent's decision, hence, achieving a price equilibrium requires computing the pay-off of each combination of pricing strategies. Furthermore, for a given pricing strategy, calculating the pay-off of a firm requires estimating the costs of their operations, which in turn imposes to solve a LSP. Thus, the complexity of computing an equilibrium may be really large as the number of periods increases, which is explained by both the complexity of the LSP and the fact that the number of pricing strategies that the competitors can select increases exponentially with the number of periods considered in the LSP.

The main objective of this work is to propose an efficient methodology for computing an equilibrium for the pricing and operations decisions between firms whose products compete in terms of their prices. Our methodology consists of two steps. In the first step, we aim to identify dominated pricing strategies, in such way they are ruled out as a potential equilibrium. We propose a Mixed Integer Programming (MIP) formulation for identifying dominated strategies. The goal of the second step is to obtain a pure Nash Equilibrium (NE) between the strategies of the firms. This task can be really challenging, since the number of strategies to be revised can be large and a NE may not exist. In such case, we assume that the firms randomize their price strategies, thus their pricing decisions will correspond to a Mixed Nash Equilibrium (Nash, 1950a). To this end, we implement the MIP formulation proposed by Sandholm et al. (2005) for obtaining MNE, and we adapt it for obtaining NE. One of the main advantages of the Sandholm et al. (2005) formulation is the possibility of including some metrics when achieving an equilibrium (e.g. maximizing the sum of the profits of the firms). This opens avenues for the accurate analysis of the competition between firms, because it provides bounds for the individual and joint performance.

The remainder of the Chapter is organized as follows. In Section 3.2 we discuss the literature of joint marketing and production decisions, paying special attention to problems between competitors. Section 3.3 contains the assumptions and description of the problem that we study in this work. In Section 3.4 we characterize the pricing strategies of the firms in order to establish dominance conditions, and in Section 3.5 we explain how to implement efficiently such characterization. We present computational results and managerial insights in Section 3.6, and we summarize the main contribution of this work in Section 3.7.

3.2 Literature Review

The need for stronger coordination between the decisions made by production and marketing departments is studied with special attention in the literature on both operations management and marketing. Upasani and Uzsoy (2007) classify existing research on joint production-marketing into three main streams: (1) joint price-production quantity determination, (2) promotion planning and coordination, and (3) price, capacity and lead time models. In this work we focus on the first research stream, where firms decide their price and production quantity subject to both production and marketing constraints.

We classify the works dealing with joint price-production quantity according to their time modeling: continuous time models and discrete time models.

The early efforts for modeling problems of coordinating pricing and production decisions assume continuous time. Indeed, the first work that provides a quantitative solution for the simultaneous production and pricing problem is proposed by Within (1955), where the pricing decision is included into the traditional Economic Order Quantity (EOQ) problem, by assuming that demand is price sensitive. In other words, the objective of the problem is the profit maximization by setting a unit price p and the ordering quantity Q. Cheng (1990) and Chen and Min (1994) study the multi-item EOQ problem, and they provide the Kuhn-Tucker conditions for determining the optimal solution under conditions of storage space and inventory investment limitations. Lee (1994) extends the problem by including an inventory investment budget constraint. The authors show that the computational time grows almost linearly with the problem size. Furthermore, several works propose Geometric Programming approaches for the single item joint EOQ and pricing problem, e.g. Kim and Lee (1998), Lee (1993) and Jung and Klein (2006).

Pekelman (1974) also deals with the continuous time problem, but prices and quantities can vary during the time, i.e. the decision variables are the production rate $q(t)$ and the optimal price $p(t)$. Vanthienen (1975) and Feichtinger and Hartl (1985) extend the work of Pekelman (1974) by including capacity constraints and by allowing backlog, respectively. In a similar context, Chen and Chu (2003) extend the problem for new products, whose sales rate is highly sensitive to price strategies. In particular, the authors propose a framework for adapting decisions to the frequent updates of the market information. Ray et al. (2005) study EOQ with price sensitive demands, assuming linear and log-linear demand functions. The authors suggest a counterintuitive conclusion: batch size is not necessarily monotone increasing in set-up cost. In order to include demand uncertainty, Adida and Perakis (2006) extend the nominal formulation to a robust optimization approach.

A second group of works assumes the horizon as a set of discrete periods. In such scenario the production decisions can be modeled as a LSP. Thomas (1970) studies simultaneous dynamic pricing and LSP. The author proposes a characterization of the problem that reduces the computations for pricing decisions. Several works simplify the dynamic pricing decision by assuming that a firm sets a constant price for each period of the planning horizon. Under such assumption, Kunreuther and Schrage (1973), Gilbert (1999) study the uncapacitated version of the problem and Gilbert (2000) extends this problem for multiple items sharing a common capacity. When the prices are not constant, Bhattacharjee and Ramesh (2000) propose heuristic approaches for the uncapacitated version of the problem. Martel and Gascon (1998) deal with the problem of maximizing profits when operations are uncapacitated and the unit inventory holding costs in a period is a function of the procurement decisions made in previous periods. The authors derive an $O(T^2)$ algorithm which provides a provably optimal solution of an integer linear programming formulation of the general problem. For capacitated operations, Deng and Yano (2006) characterize the optimal solution, and they propose an algorithm whose complexity is slightly higher than the traditional LSP. The authors show evidence that larger capacity may lead to higher optimal prices. Haugen et al. (2007) propose a Lagrangian relaxation procedure to solve the problem for capacitated operations. For the same problem, Gonzalez-Ramirez et al. (2011) propose a Dantzig-Wolfe decomposition where the capacity constraints are the linking constraints. Onal and Romeijn (2011) extend the capacitated problem to the multi-item case. They develop two alternative Dantzig-Wolfe decomposition formulations of the problem, and propose to solve their relaxations using column generation and the overall problem using branch-and-price. The authors test such formulations for both dynamic and static pricing strategies. Similar to our assumptions (but assuming the decisions of a firm in isolation), Lanquepin-Chesnais et al. (2012) study a single item problem where a firm maximizes its profit over a discrete set of prices. The authors propose a Lagrangian relaxation to solve this problem.

Despite the large literature about simultaneous production and pricing decisions, the discussion of making such decisions in competitive environments has received less attention. Min (1992), Chen et al. (1995) are among the earliest works that address the simultaneous production and pricing problem under competition. The authors model the operations of the firms as an EOQ problem, where sellers of a homogeneous product compete within each other for the same buyers. Cachon and Harker (2002) study the competition between two firms whose production systems are characterized by economies of scale, such as in the case of the EOQ model. The authors derive general conditions under which there exists at most one equilibrium between competitors. Transchel and Minner (2011) propose an extension of the problem where a firm has an EOQ cost structure, and its opponent follows a just-in-time policy. Adida and Perakis (2010) address the problem where two firms compete through dynamic pricing and inventory control with the presence of demand uncertainty. The authors introduce a demand base fluid model where the demand is a linear function of the price of the supplier and of her competitor, the inventory and production costs are quadratic, and all coefficients are time dependent.

More related to our work, several papers discuss competition in discrete time. Fredergruen and Meissner (2009) study the competition between firms when their production lines are modeled as a capacitated LSP. The authors establish the existence of a price equilibrium and associated optimal dynamic lot sizing plans, under mild conditions. In contrast to our work, they assume that each firm sets a constant price for the whole interval of production. The constant price condition is relaxed in Pedroso and Smeers (2010). The authors propose a fix-point iteration for achieving a duopoly equilibrium when competitors produce perfect substitute items. The main differences with our work are the following: we assume that the firms select prices from a discrete set, we examine the possibility of randomizing price strategies (MNE), and the competition may occur between non-perfect substitutes. To the best of our knowledge, this is the first work including such dimensions for discrete time problems.

3.3 Problem Description

Consider two self-interested firms, Firm 1 and Firm 2, that manufacture and sell mutually substitute products. Because the existence of a substitution effect, the demands for these products depend on the price set by both firms. Each firm, therefore, maximizes its profit by making pricing decisions, which should take both the cost of the operations and the opponent's price into account.

We assume that each firm organizes its production by setting a plan for N consecutive periods. The problem of making pricing and production decisions constitutes a repeated game, such that in each repetition the firms make decisions for N periods. The dynamics in each repetition of the game are as follows. In our description Firm i symbolizes any of Firm 1 or Firm 2, and Firm j corresponds to its opponent.

- Before starting the production activities for the next N periods, the firms decide the unit price of their items for each of such periods. For each period, Firm i can select a price from the discrete set $\{L_i, M_i, U_i\}$, where $0 < L_i < M_i < U_i$. These prices represent aggressive strategies (L_i), medium prices (M_i) or conservative prices (U_i). Let $s_i \in \mathbf{R}^N$ be the vector containing the price decision of firm i such that $s_{i,n} \in \{L_i, M_i, U_i\}$. We call this vector a pure pricing strategy. Further, we denote S_i to the set of all pure pricing strategies of the firm i. Note that, $|S_i| = 3^N$, then the number of strategies that the firms can select grows exponentially when the number of periods increases.

- The pure pricing strategies selected by both firms induce their demands for the next N periods. We denote by $d_i : S_i \times S_j \to \mathbf{R}^N$ the mapping between prices and demands of Firm i. We construct this mapping assuming that the demand is a linear function of the prices, as we show in equation (3.1).

$$d_{i,n}(s_i, s_j) = \alpha_{i,n} - \beta_{i,n} \cdot s_{i,n} + \gamma_{i,n} \cdot s_{j,n}, \quad \forall i = 1,2, j \neq i, \tag{3.1}$$

  $\alpha_{i,n}$ represents the demand of Firm i when the prices of both firms are equal to 0 in the period n. We can model the seasonality of the demand by assuming that this parameter is time dependent; $\beta_{i,n}$ is the sensitivity of the demand of Firm i to the price of its own product in period n. We assume there is an inverse relation between prices and demand of the firm; and $\gamma_{i,n}$ is the sensitivity of the demand of Firm i to the price of the item produced by firm j in period n. As the products of the firms are substitutes, we establish that $\gamma_{i,n} \geq 0$.

- Once the firms made their pricing decisions, they realize their demands for the next N periods. Based on this information, each firm organizes its operations with the aim to minimize its costs subject to the demand satisfaction. We model the operations of each firm in a planning block as a *LSP*.

Further, we assume that each firm has perfect information about the set of prices that its opponent can choose. This implies that the pure pricing strategies that firms will select constitutes a NE. Nevertheless, given that the set of prices is discrete, a NE may not exist. In such a case, we assume that firms randomize their pricing decisions, i.e. firm i will select a pure pricing strategy $s_i \in S_i$ with probability a certain probability. The probabilities that firms assign to each pure pricing strategy constitute a MNE.

In the remaining part of this section we describe the modeling of the decisions related to the operations and pricing of the firms, and we formulate NE and MNE as sets of linear MIP constraints.

3.3.1 Operations Planning

Once the firms choose the pure pricing strategies $s_i \in S_i$ and $s_j \in S_j$, each of them plan their operations by solving a LSP, in such a way demand is satisfied at minimum cost. The parameters of the LSP faced by each firm are $c, f, h \in \mathbf{R}^{2\times N}$ and $CAP \in \mathbf{R}^2$, where:

- $c_{i,n}$: unit production cost of the item sold by firm i in period n.
- $f_{i,n}$: set-up cost of the line managed by firm i in period n.
- $h_{i,n}$: unit holding cost of the item sold by firm i in period n.
- CAP_i : production capacity per period of firm i.
- $d_{i,n}(s_i, s_j)$: demand faced by the firm i in period n (see equation (3.1)).

The decisions variables of the *LSP* are $x, y, z \in \mathbf{R}^{2\times N}$, such that:

- $x_{i,n}$: amount of production of item i in period n.
- $y_{i,n}$: 1 if the line managed by firm i is used in period n, 0 otherwise.
- $z_{i,n}$: amount of inventory of the item held by firm i at the end of period n.

The set of constraints that ensure that the operations of Firm i satisfies the demand derived from pure pricing strategies s_i and s_j are as follows:

$$z_{i,n} = z_{i,n-1} + x_{i,n} - d_{i,n}(s_i, s_j) \quad \forall n \in N \tag{3.2}$$

$$x_i \leq CAP_i \cdot y_i, \tag{3.3}$$

$$x_i, z_i \geq 0, y_i \in \{0,1\}, \tag{3.4}$$

where, constraint (3.2) represents the inventory conservation and demand satisfaction; constraint (3.3) states that production in a period can occur only if the production in such period is activated; constraint (3.3) also reflects the capacity limitation in each period; if the problem is uncapacitated we replace CAP_i by a big number. We denote by $\mathscr{X}_i(s_i,s_j)$ to the set of all the decisions x_i that satisfy the constraints (3.2) - (3.4) when the firms play the pair of pure pricing strategies (s_i,s_j).

Furthermore, we say that the function $C_i(x)$ provide the total operational cost of Firm i when the firms play the pair of pure pricing strategies (s_i,s_j), where

$$C_i(x) = c_i \cdot x_i^T + f_i \cdot y_i^T + h \cdot z_i^T. \tag{3.5}$$

3.3.2 Pricing

When making pricing decisions, each firm expects to maximize its profit, which corresponds to the difference between its incomes and its operational costs. If the firms select the pure pricing strategies $(s_i,s_j) \in S_i \times S_j$ for the next N periods, we represent the incomes, operational costs and profit by the function $R : S_i \times S_j \to \mathbf{R}^2$, $C^o : S_i \times S_j \to \mathbf{R}^2$ and $B : S_i \times S_j \to \mathbf{R}^2$, respectively, where $R_i(s_i,s_j) = s_i \cdot d_i(s_i,s_j)$, $C_i^o(s_i,s_j) = \min_{x_i} \{C_i(x_i) | x_i \in \mathscr{X}_i(s_i,s_j)\}$ and $B_i(s_i,s_j) = R_i(s_i,s_j) - C_i^o(s_i,s_j)$.

Here, we describe the problem of selecting pure pricing strategies and the problem of randomizing pure pricing strategies.

Pure pricing strategies

Taking the pricing decision of its opponent into account, each firm makes its pricing decision in order to maximize its own profit, i.e.

$$\begin{array}{ll} \mathbf{max}_{s_i} & B_i(s_i,s_j) \\ \text{s.t.} & (s_i,s_j) \in S_i \times S_j \end{array} \tag{3.6}$$

Since the pricing decisions of the firms are mutually dependent, the firms should choose pure pricing strategies that satisfy the condition of a NE. The pair (s_i,s_j) is a NE if $B_i(s_i,s_j) \geq B_i(s_i',s_j), \forall s_i' \in S_i$ and $\forall i = 1,2; j \neq i$. In other words, a firm does not have an incentive to change its pure pricing strategy given that the strategy of its opponent has not changed.

Randomizing pure pricing strategies

When the firms randomize their pure pricing strategies, the decision for each of them is to assign a probability to each pure pricing strategy, in such way it maximizes its own profit. Let $p_i \in [0,1]^{|S_i|}$ be the vector containing the probabilities that Firm i assigned to its pure pricing strategies in a repetition of the game. We formulate the pricing problem of the Firm i by model (3.7).

$$\begin{aligned}
\mathbf{max}_{p_i} \quad & \mathbf{E}_{p_i,p_j}[B_i(s_i,s_j)] \\
\text{s.t.} \quad & (s_i,s_j) \in S_i \times S_j \\
& \sum_{s_i \in S_i} p_i(s_i) = 1 \\
& p_i \geq 0 \quad \forall s_i \in S_i
\end{aligned} \tag{3.7}$$

Given that the probability assignment are mutually dependent, the solution of the pricing problem should satisfy the condition of a MNE. (p_i, p_j) is a MNE if $\sum_{s_i \in S_i} p_{i,s_i} \cdot \sum_{s_j \in S_j} p_{j,s_j} \cdot B_i(s_i,s_j) \geq \sum_{s_i \in S_i} p'_{i,s_i} \cdot \sum_{s_j \in S_j} p_{j,s_j} \cdot B_i(s_i,s_j), \forall p'_i$ and $\forall i = 1,2; j \neq i$. In other words, a firm does not have an incentive to change its mixed strategy given that the mixed strategy of its opponent has not changed. Nash (1950b) proofs that there is at least one equilibrium for this game.

Nash Equilibrium and Mixed Nash Equilibrium

We characterize the pricing strategies that satisfy the NE and MNE condition by utilizing MIP formulations. These formulations are based on two facts: *in any equilibrium, every pure strategy is either played with probability 0, or has 0 regret*; and, *any vector of mixed strategies for the players where every pure strategy is either played with probability 0, or has 0 regret, is an equilibrium*. Sandholm et al. (2005) propose the set of MIP constraints of expressions (3.8) - (3.13) to describe a MNE. This formulation utilizes two parameters: $\pi \in \mathbf{R}^{2\times|S_i|\times|S_j|}$ such that $\pi_{i,s_i,s_j} = B_i(s_i,s_j)$ and $\bar{\pi} \in \mathbf{R}^2$, such that $\bar{\pi}_i = \mathbf{max}_{s_i^l,s_i^h,s_j^l,s_j^h \in S}(\pi_{i,s_i^h,s_j^h} - \pi_{i,s_i^l,s_l^h})$. Although, the decision variables are the vector of probabilities $p_i, \forall i = 1,2$, the authors utilize additionally the variables $b_i, r_i \in \mathbf{R}^{|S_i|}$, $u_i \in \{0,1\}^{|S_i|}$ and $\bar{b}_i \in \mathbf{R}$, where:

- b_{i,s_i} : expected profit of Firm i when playing strategy $s_i \in S_i$.
- r_{i,s_i} : expected regret when the firm i when playing the strategy $s_i \in S_i$.
- u_{i,s_i} : 1 if Firm i plays the strategy $s_i \in S_i$ with probability 0; 0 otherwise.
- $\bar{b}_i$: Expected profit of Firm i.

$$\sum_{s_i \in S_i} p_{i,s_i} = 1, \quad \forall i = 1,2, \tag{3.8}$$

$$b_i = \pi_i \cdot p_j^T \quad \forall i = 1,2; j \neq i, \tag{3.9}$$

$$r_{i,s_i} = \bar{b}_i - b_{i,s_i} \quad \forall s_i \in S_i; \forall i = 1,2, \tag{3.10}$$

$$p_{i,s_i} \leq 1 - u_{i,s_i} \quad s_i \in S_i; \forall i = 1,2, \tag{3.11}$$

$$r_{i,s_i} \leq \bar{\pi}_i \cdot u_{i,s_i} \quad \forall s_i \in S_i; \forall i = 1,2, \tag{3.12}$$

$$p_i, b_i, \bar{b}_i, r_i \geq 0; u_i \in \{0,1\} \quad \forall i = 1,2. \tag{3.13}$$

Constraint (3.8) ensures that p_i is a valid probability distribution. Constraints (3.9) - (3.10) define the regret of a strategy. Constraint (3.11) ensures that $u_{i,s} = 1$ only if $p_{s,i} = 0$. Constraint (3.12) states that the regret of a strategy equals 0, unless the strategy is played with probability 0.

Even though this formulation aims to characterize a MNE, its extension to to NE is straightforward. A NE characterization is obtained by including the constraint $p_i \in \{0,1\}, \forall i = 1,2$ into the previous formulation.

Another strength of this characterization lies on the possibility to combine constraints (3.8) - (3.12) with an objective function that represents an specific goal of the equilibrium between firms. For instance, the objective function $\mathbf{max}(\overline{b}_i + \overline{b}_j)$ will lead to the equilibrium that maximizes the sum of the profit of the firms, on the contrary, objective function $\mathbf{min}(\overline{b}_i + \overline{b}_j)$ will correspond to the equilibrium that minimizes such sum. Other objective functions can provide the minimization of the profit of a firm ($\mathbf{min}(\overline{b}_i)$) or the maximization of such profit ($\mathbf{max}(\overline{b}_i)$). Moreover, we can achieve a pricing equilibrium that leads a balance between the profits of the firms ($\mathbf{maxmin}\{\overline{b}_i, \overline{b}_j\}$).

However, achieving a NE or a MNE by this formulation may be intensive in computational requirements. The high complexity arises from the fact that characterization (3.8) - (3.12) takes the full payoff matrix B as a parameter, which implies solving $2 \cdot 3^{2N}$ different *LSP*s. Moreover, such formulation consists of $2 \cdot (3^{N+1} + 1)$ continuous variables and $2 \cdot 3^N$ binary variables, hence, the dimension of the problem increases exponentially with N.

We propose a methodology to deal with the exponential complexity for computing an equilibrium between the pricing decisions of the firms. This methodology identifies and discards dominated strategies from the set of potential equilibria. We can discard a strategy because a NE or a MNE can be calculated by comparing uniquely the best responses of the firms. So, achieving a NE or a MNE will requires to compute partially the pay-off matrices of the firms. In Section 3.4 we describe and analyze the problem of finding the best response to each strategy of the opponent.

3.4 Best Response Problem

We start by defining the best response of a firm to a pure pricing strategy of its opponent.

Definition 3.1. *$s_i^*(s_j) \in S_i$ is the best response of the firm i to $s_j \in S_j$, such that $B_i(s_i^*(s_j), s_j) \geq B_i(s_i, s_j), \forall s_i \in S_i$.*

Our aim is to provide a MIP formulation for the finding $s_i^*(s_j)$. With this in mind, we introduce the decision variable $w_i \in \{0,1\}^{3\times N}$, where $w_{i,k,n}$ is 1 if the firm i selects the price $k \in \{L_i, M_i, U_i\}$ for the period n; 0 otherwise. Thus, we can express any $s_i \in S_i$ as a linear combination of w (as we show in equation (3.14)), and consequently, the best response problem to any pricing strategy of the opponent can be formulated by the MIP formulation (3.15) (3.17).

$$s_i(w_i) = L_i \cdot w_{i,L_i} + M_i \cdot w_{i,M_i} + U_i \cdot w_{i,U_i} \tag{3.14}$$

$$\begin{aligned}
\mathbf{max}_w \quad & R_i(s_i(w), s_j) - C_i^o(s_i(w), s_j) & (3.15)\\
\text{s.t.} \quad & w_{i,L_i,n} + w_{i,M_i,n} + w_{i,U_i,n} = 1, \quad \forall n = 1,...,N, & (3.16)\\
& w_{i,L_i,n}, w_{i,M_i,n}, w_{i,U_i,n} \in \{0,1\}, \quad \forall n = 1,...,N. & (3.17)
\end{aligned}$$

Expression (3.15) states the maximization of the profit of the firm. Note that, $C_i^o(s_i(w), s_j)$ is the cost obtained by Firm i when solving the *LSP* resulting of choosing strategies $s_i(w)$ and s_j. Constraints (3.16) - (3.17) states that Firm i must select one and only one price per period.

Alternatively to the full computation of payoff for calculating an equilibrium between firms, we can compute the payoff of the best responses of the firms based on model (3.15) - (3.17). Hence, we need to solve only $2 \cdot 3^N$ best response models, which is less than the $2 \cdot 3^{2 \cdot N}$ problems required for a full computation of the payoff. Moreover, even though model (3.15) - (3.17) consists of $3 \cdot T$ new binary variables for each *LSP*, solving the problem may be faster than the full computation of payoff. This is because the full computation is an iterative procedure that does not consider the structure of the problem, but model (3.15) - (3.17) can catch such structure by implementing simple well known algorithms, such as Branch & Bound or Branch & Cut.

Furthermore, we can reduce the time for calculating an equilibrium by determining a priori the characteristics of a firm that lead to prefer a specific pure pricing strategy as a best response to a strategy of its opponent. This reduction stems from two observations: first, when calculating the best response to s_j, we could establish beforehand the pure pricing strategies of Firm i that will not be selected as a best response to an opponent's strategy; second, when we calculate sequentially the elements of the matrix of payoff of a firm, we can utilize the already calculated equilibrium to characterize the best response of a firms to its opponent's strategy, so we reduce the number of pure pricing strategies to be revised.

Our characterization of $s_i^*(s_j)$ is obtained by bounding the effects of small variations of the pure pricing strategies of the opponent on the operations of a firm. In this characterization, we assume that the unit production cost of a firm does not vary with the time, i.e., $c_{i,n} = c_i, \forall n = 1,...,N$. Before describing our characterization, we provide some definitions that facilitate the exposition of our analysis.

Definition 3.2. *Given a pure pricing strategy s_i, $\delta_{\hat{n}}(s_i) = \left\{s_i' | s_i' \in S_i \wedge s_{i,n}' = s_{i,n}, \forall n \neq \hat{n} \wedge s_{i,\hat{n}}' \geq s_{i,\hat{n}}\right\}$, i.e., $\delta_{\hat{n}}(s_i)$ is the set of pure pricing strategies that increase the price in period $\hat{n}$ respect to s_i.*

Definition 3.3. *Given the pure pricing strategy $s_j \in S_j$, $\lambda(s_i', s_i, s_j)$ is the difference between the optimal cost of operations of Firm i when it plays the strategies s_i and s_i', i.e. $\lambda(s_i', s_i, s_j) = C_i^o(s_i, s_j) - C_i^o(s_i', s_j)$.*

Definition 3.4. *Given the pure pricing strategy $s_i \in S_i$, $\tau(s_j', s_i, s_j)$ is the difference between the optimal cost operations of firm i when firm j plays strategies s_j and s_j', i.e. $\tau(s_i, s_j', s_j) = C_i^o(s_i, s_j') - C_i^o(s_i, s_j)$.*

Additionally, we will utilize $\phi(s_j, n) = \frac{\alpha_{i,n} + \gamma_{i,n} \cdot s_{j,n}}{\beta_{i,n}}$, $\Delta(s_i', s_i) = \sum_{n=1}^{N}(s_{i,n}' - s_{i,n})$ and $n^*(s_j) = \mathbf{min}\{n | d_{i,n}(U_i, s_j) > 0\}$, where $\phi(s_j, n)$ is the maximum price that Firm i should set in order to obtain non negative demand in period n, given the opponent's strategy s_j; $\Delta(s_i', s_i)$ corresponds to the sum of the different of prices between the the pure pricing strategies s_i' and s_i. Note that, $\Delta(s_i', s_i) = s_{i,\hat{n}}' - s_{i,\hat{n}}$ if $s_i' \in \delta_{\hat{n}}(s_i)$; and $n^*(s_j)$ is the earliest period in which we can ensure that the demand of Firm i is non-negative given the opponent's strategy s_j.

3.4.1 Ex-ante Characterization

Given a certain strategy of the opponent, our characterization aims to determine the choice of the firm between two of its pure pricing strategies. This choice depends on the condition (3.18), i.e. Firm i will choose s_i instead of s_i' as a response to s_j if the additional revenues obtained through s_i is larger than the additional operational costs associated with such strategy.

$$R_i(s_i, s_j) - R_i(s_i', s_j) > \lambda_i(s_i', s_i, s_j). \tag{3.18}$$

Although writing a closed form expression for the left hand side of equation (3.18) is straightforward, computing the right hand side implies solving two *LSP*, thus we cannot express such difference by a closed form function. Further, solving the two *LSP* problems implies a high complexity as the value of n increases, that we would like to avoid in our characterization. Hence, we focus on obtaining closed form estimators for the difference between the costs associated to two pure pricing strategies, in such a way, we can utilize expression (3.18) to compare such strategies.

Uncapaciated operations

We start our characterization by providing upper and lower bounds for $\lambda(s'_i, s_i, s_j)$ when $s'_i \in \delta_{\hat{n}}(s_i)$, as we show in Lemma 3.1.

Lemma 3.1. *Given the pure pricing strategies $s_i \in S_i$ and $s'_i \in \delta_{\hat{n}}(s_i)$, $\underline{\lambda}(s'_i, s_i)$ and $\overline{\lambda}(s'_i, s_i, s_j)$ are a lower bound and an upper bound for $\lambda(s'_i, s_i, s_j)$, respectively, where*

$$\underline{\lambda}(s'_i, s_i) = c_i \cdot \beta_{i,\hat{n}} \cdot \Delta(s'_i, s_i),$$

$$\overline{\lambda}(s'_i, s_i, s_j) = \begin{cases} \underline{\lambda}(s'_i, s_i) + h_i \cdot (\hat{n} - n^*(s_j)) \cdot \beta \cdot \Delta(s'_i, s_i), & \textit{if } n^*(s_j) \leq \hat{n} \\ \underline{\lambda}(s'_i, s_i) + f_i, & \textit{otherwise,} \end{cases}$$

and

$$n^*(s_j) = \mathbf{min}\{n | d_{i,n}(U_i, s_j) > 0\}.$$

Proof. Given that $d_{i,n}(s'_i, s_j) = d_{i,n}(s_i, s_j), \forall n \neq \hat{n}$ and $d_{i,\hat{n}}(s'_i, s_j) \leq d_{i,\hat{n}}(s_i, s_j)$, if firm i plays s'_i instead of s_i, then its total production cost will decrease by at least $c_i \cdot (d_{i,\hat{n}}(s_i, s_j) - d_{i,\hat{n}}(s'_i, s_j))$. By replacing equation (3.1) in the right hand side of the last expression we obtain $\underline{\lambda}(s'_i, s_i)$.

We should also estimate the changes in the total set-up and holding costs. Let us assume that we know the periods in which Firm i carries out its production in order to minimize its cost when it plays s'_i. In other words we utilized the vector $y^*_i(s'_i, s_j) \in \{0,1\}^N$ which is obtained by minimizing expression (3.5) subject to the conditions (3.2)-(3.4) for the pair (s'_i, s_j). We can analyze three scenarios:

- if $y^*_{i,\hat{n}}(s'_i, s_j) = 1$, we can construct an operations plan for s_i, whose total set-up and holding costs are equal to the ones obtained for s'_i. This corresponds to a best case scenario, thus $\underline{\lambda}(s'_i, s_i) \leq \lambda(s'_i, s_i, s_j)$.
- if $y^*_{i,\hat{n}}(s'_i, s_j) = 0$, we again can construct a feasible operations plan for s_i based on $y^*_i(s'_i, s_i)$, but the additional demand caused by s_i has to be produced in the interval $[1, \hat{n} - 1]$. When $n^*(s_j) < \hat{n}$, we certainty know that there is at least one period of the interval $[1, \hat{n} - 1]$ in which the production is carried out. If the additional demand caused by s_i is satisfied from period $n^*(s_j)$, the resulting costs

will lead an upper bound for $\lambda(s'_i, s_i, s_j)$. Hence, the holding cost is increased in $h_i \cdot (\hat{n} - n^*(s_j)) \cdot (d_{i,\hat{n}}(s_i, s_j) - d_{i,\hat{n}}(s'_i, s_j))$, but no additional setup occurs.

- $y^*_{i,\hat{n}}(s'_i, s_j) = 0$ and $n^*(s_j) > \hat{n}$, we cannot ensure that the demand in the interval $[1, \hat{n} - 1]$ is different to zero. In this scenario, we can obtain an upper bound for $\lambda(s'_i, s_i, s_j)$ by activating the production in period $\hat{n}$. Hence, an additional set-up is required, but the total holding does not vary with respect to the operations for s'_i.

□

We use $\underline{\lambda}(s'_i, s_i)$ to determine whether s_i or s'_i is a better response to a certain strategy of the opponent, as we show in Theorem 3.1.

Theorem 3.1. *Given the pure pricing strategies $s_i \in S_i$ and $s'_i \in \delta_{\hat{n}}(s_i)$, if $s_{i,\hat{n}} + s'_{i,\hat{n}} < \phi(s_j, \hat{n}) + c_i$, then s'_i is preferable to s_i as a response to s_j.*

Proof. From expression (3.18), s'_i is preferable to s_i as a response to s_j if $R_i(s'_i, s_j) - R_i(s_i, s_j) > -\lambda(s'_i, s_i, s_j)$. Since the right hand side of the previous expression is upper bounded by $-\underline{\lambda}(s'_i, s_i)$, $R_i(s'_i, s_j) - R_i(s_i, s_j) > -\underline{\lambda}(s'_i, s_i)$ is a sufficient condition for discarding s_i as a best response to s_j. By replacing equation (3.1) in the previous expression, we obtain the condition presented in Theorem 3.1. □

Form Theorem 3.1 we derive conditions for selecting or discarding prices U_i and L_i as part of a best response strategy, as we show in Corollary 3.1.

Corollary 3.1. *Given that Firm j chooses the pure pricing strategy $s_j \in S_j$,*

(a) if $U_i + M_i \leq \phi(s_j, n) + c_i$, then $s^*_{i,n}(s_j) = U_i$,

(b) if $M_i + L_i \leq \phi(s_j, n) + c_i$, then $s^*_{i,n}(s_j) \neq L_i$.

Proof. We provide a proof for the part (a) of this corollary. The proof for the part (b) is analogous. We analyze the condition $s_{i,\hat{n}} + s'_{i,\hat{n}} \leq \phi(s_j, \hat{n}) + c_i$ introduced in Theorem 3.1. Given that the left hand side of such condition increases with the value of the prices, the maximum of such sum occurs for $s'_{i,\hat{n}} = U_i$ and $s_{i,\hat{n}} = M_i$. Thus, $U_i + M_i \leq \phi(s_j, n) + c$ is a sufficient condition for preferring U_i. □

In a similar fashion as Theorem 3.1, we utilize $\overline{\lambda}(s'_i, s_i, s_j)$ to analyze the convenience of setting a specific price as part of the best response to the opponent's strategy, as we show in Theorem 3.2.

Theorem 3.2. *Given the pure pricing strategies $s_i \in S_i$ and $s'_i \in \delta_{\hat{n}}(s_i)$, if $s_{i,\hat{n}} + s'_{i,\hat{n}} \geq \frac{\overline{\lambda}(s'_i, s_i, s_j)}{\beta_{i,n} \cdot \Delta(s'_i, s_i)} + \phi(s_j, \hat{n})$, then s_i is preferable to s'_i as a response to s_j.*

Proof. From expression (3.18), if $R_i(s_i,s_j) - R_i(s'_i,s_j) \geq \lambda(s'_i,s_i,s_j)$, then s_i is preferable to s'_i as a response to s_j. Given that the right hand side of the previous expression is upper bounded by $\overline{\lambda}(s'_{i,\hat{n}},s_{i,\hat{n}},s_j,\hat{n})$, $R_i(s_i,s_j) - R_i(s'_i,s_j) \geq \overline{\lambda}(s'_i,s_i,s_j)$ is a sufficient condition for discarding s'_i as a best response to s_j. By replacing equation (3.1) in the previous expression, we obtain the condition presented in Theorem 3.2. $\square$

Based on Theorem 3.2, we derive additional conditions for selecting or discarding prices U_i and L_i, as we show in Corollary 3.2.

Corollary 3.2. *Given that Firm j plays the pure pricing strategy $s_j \in S_j$,*

(a) if $n^*(s_j) \leq n \leq n^*(s_j) + \frac{L_i+M_i-c_i-\phi(s_j,n)}{h_i}$ or $L_i + M_i - c_i - \phi(s_j,n) \geq \frac{f_i}{\beta_{i,n}\cdot(M_i-L_i)}$, then $s^*_{i,n}(s_j) = L_i$,

(b) if $n^*(s_j) \leq n \leq n^*(s_j) + \frac{M_i+U_i-c_i-\phi(s_j,n)}{h_i}$ or $M_i + U_i - c_i - \phi(s_j,n) \geq \frac{f_i}{\beta_{i.n}\cdot(U_i-M_i)}$, then $s^*_{i,n}(s_j) \neq U_i$.

Proof. We start by proving the part (a) of this corollary. In particular, we are interested in the condition $n^*(s_j) \leq n \leq n^*(s_j) + \frac{L_i+M_i-c_i-\phi(s_j,n)}{h_i}$, since the rest of the conditions introduced in the corollary can be derived in a similar way. If we replace the value of $\overline{\lambda}(s'_i,s_i,s_j)$ of Lemma 3.1 in the condition presented in Theorem 3.2, we obtain $s_{i,\hat{n}} + s'_{i,\hat{n}} \geq c_i + h_i \cdot (\hat{n} - n^*(s_j)) + \phi(s_j,\hat{n})$. Given that the minimum value of $s_{i,\hat{n}} + s'_{i,\hat{n}}$ is $L_i + M_i$ and $\phi(s_j,\hat{n})$ is increasing with $s_{j,\hat{n}}$, the analyzed expression is a sufficient condition for preferring L_i. $\square$

We can utilize the results of Corollary 3.1 and Corollary 3.2 to select and discard prices from the best response strategy of a firm. In particular, Corollary 3.1 provides bounds for $s^*_{i,n}(s_j)$, that depend only on the demand function and the prices of the firms. Corollary 3.2 derives upper bounds for $s^*_{i,n}(s_j)$, but such bounds are time dependent.

Capacitated operations

As for the uncapacitated operations, we start by providing a closed expression for upper and lower bounds of $\lambda(s'_i,s_i,s_j)$. Note that, for capacitated operations we assume that the capacity per period is larger than or equal to the maximum demand that a firm can face in a period, i.e. $CAP_i \geq \alpha_{i,n} - \beta_{i,n}\cdot L_i + \gamma_{i,n}\cdot U_j$. Thus, all demand can be satisfied on time.

Lemma 3.2. *Given the pure pricing strategies $s_i \in S_i$ and $s'_i \in \delta_{\hat{n}}(s_i)$, $\underline{\lambda}(s'_i,s_i)$ and $\overline{\lambda}_C(s'_i,s_i,s_j)$ are a lower bound and an upper bound for $\lambda(s'_i,s_i,s_j)$, respectively, where*

$$\overline{\lambda}_C(s'_i,s_i,s_j) = \underline{\lambda}(s'_i,s_i) + h_i\cdot(\hat{n}-1)\cdot\beta_{i,n}\cdot\Delta(s'_i,s_i) + f_i,$$

Proof. If Firm i chooses s_i instead of s'_i, then the total unit production cost will increase in at least $c_i \cdot (d_{i,\hat{n}}(s_i,s_j) - d_{i,\hat{n}}(s'_i,s_j))$. By replacing equation (3.1) in the right hand side of the last expression, we obtain $\underline{\lambda}(s'_i,s_i)$. In an optimistic case, Firm i will not incur in additional set-up or holding cost, thus, $\underline{\lambda}(s'_i,s_i) \leq \lambda(s'_i,s_i,s_j)$.

To estimate $\overline{\lambda}_C(s'_i,s_i,s_j)$ we construct a feasible plan where Firm i satisfies the additional demand in $\hat{n}$ by manufacturing in the interval $[1,\hat{n}]$. We obtain an upper bound for the holding costs of satisfying the additional demand by assuming that such demand is served from the period 1. The worst case in terms of the set-up costs is to initialize one more period for production. Thus, $\overline{\lambda}_C(s'_i,s_i,s_j) \geq \lambda(s'_i,s_i,s_j)$. □

Given that $\underline{\lambda}(s'_i,s_i) = \underline{\lambda}_C(s'_i,s_i)$, Theorem 3.1 and Corollary 3.1 are also valid for capacitated LSP. Nevertheless, we cannot extend directly Corollary 3.2 to capacitated operations. To this end, we replace $\overline{\lambda}(s'_i,s_i,s_j)$ by $\overline{\lambda}_C(s'_i,s_i,s_j)$ in Theorem 3.2. The resulting corollary is as follows.

Corollary 3.3. *Given the pure pricing strategies $s_i \in S_i$ and $s'_i \in \delta_{\hat{n}}(s_i)$ and Firm j chooses the pure pricing strategy $s_j \in S_j$,*

(a) if $h_i \cdot (n-1) \leq L_i + M_i - c_i - \phi(s_j,n) - \frac{f_i}{\beta_{i,n} \cdot \Delta(s'_i,s_i)}$, then $s^*_{i,n}(s_j) = L_i$,

(b) if $h_i \cdot (n-1) \leq M_i + U_i - c_i - \phi(s_j,n) - \frac{f_i}{\beta_{i,n} \cdot \Delta(s'_i,s_i)}$, then $s^*_{i,n}(s_j) \neq U_i$.

3.4.2 Characterization Based on Other Best Responses

Now, we characterize $s^*_i(s_j)$ based on the best response to another pure pricing strategy of the opponent. By this characterization we establish conditions for setting the prices per period of the best response strategies. We provide the conditions for the scenario in which the firms work under the uncapacitated LSP. So, our objective is to provide the conditions that ensures that Firm i will prefer the strategy s_i as a response to $s'_j \in \delta_{\hat{n}}(s_j)$, given that s_i is the best response to s_j. To elucidate such preference, we start by proposing closed form expressions for upper and lower bounds of $\tau(s'_j,s_i,s_j)$ in Lemma 3.3.

Lemma 3.3. *Given the pure pricing strategies $s_j \in S_j$ and $s'_j \in \delta_{\tilde{n}}(s_j)$, $\underline{\tau}(s'_j,s_j)$ and $\overline{\tau}(s'_j,s_i,s_j)$ are a lower and an upper bounds for $\tau(s'_j,s_i,s_j)$, respectively, where*

$$\underline{\tau}(s'_j,s_j) = c_i \cdot \gamma_{i,n} \cdot \Delta(s'_j,s_j),$$

$$\overline{\tau}(s'_j,s_i,s_j) = \underline{\tau}(s'_j,s_j) + \mathbf{min}\left\{\overline{h}_i(s_i,s_j) \cdot \gamma_{i,n} \cdot \Delta(s'_j,s_j), f_i\right\},$$

and

$$\overline{h}_i(s_i,s_j) = h_i \cdot (\tilde{n} - \mathbf{max}_{n=1,\dots,\tilde{n}}\{n | y^*_{i,n}(s_i,s_j) = 1\}),$$

Proof. Given that $d_{i,n}(s_i,s_j) = d_{i,n}(s_i,s'_j), \forall n \neq \tilde{n}$ and $d_{i,\tilde{n}}(s_i,s_j) \leq d_{i,\tilde{n}}(s_i,s'_j)$, if the opponent changes its strategy from s_j to s'_j, the total unit production cost of Firm i will increase by at least $c_i \cdot (d_{i,\tilde{n}}(s_i,s'_j) - d_{i,\tilde{n}}(s_i,s_j))$. By replacing equation (3.1) in the right hand side of the last expression we obtain $\underline{\tau}(s'_j,s_j)$.

In order to estimate the changes in the total set-up and holding costs, we utilize the information contained in $y^*_i(s_i,s_j)$. We analyze three scenarios:

- if $y^*_{i,\tilde{n}}(s_i,s_j) = 1$, firm i can construct an operations plan for s'_j whose total set-up and holding costs do not change with respect to the optimal plan for s_j. This is a best case scenario, thus $\underline{\tau}(s'_j,s_j) \leq \tau(s_i,s'_j,s_j)$,
- if there is at least one setup in the interval $[1,\tilde{n}]$ when Firm i satisfies the demand derived from (s_i,s_j), then Firm i can manufacture the additional due to s'_j without additional setups. In order to minimize costs, such additional demand should be served from the last period of the interval $[1,\tilde{n}]$ in which the production is activated, i.e. $\max_{n=1,\ldots,\tilde{n}}\{n|y^*_{i,n}(s_i,s_j) = 1\}$. Thus, the total holding cost increases in $\overline{h}_i \cdot \gamma_{i,\hat{n}} \cdot \Delta(s'_j,s_j)$, but no additional setup occurs,
- if the optimal operations plan of Firm i related to the pair (s_i,s_j) does not lead to produce in the interval $[1,\tilde{n}]$, Firm i can satisfy the additional demand by activating the production in $\tilde{n}$. Hence, a new setup cost arises, but no additional holding cost exists.

The minimum between the values of the last two scenarios provides the upper bound for $\tau(s_i,s'_j,s_j)$ proposed in the lemma. □

From Lemma 3.3 we can use the price information of $s^*_i(s_j)$ to characterize $s^*_i(s'_j)$ when $s_j \in \delta_{\tilde{n}}(s_j)$. In particular, we obtain upper and lower bounds for the price of the best response in each period, as we show in Theorem 3.3.

Theorem 3.3. *Given the pure pricing strategies $s_j \in S_j$ and $s'_j \in \delta_{\tilde{n}}(s_j)$, and $s_i \in S_i$ such that $s_{i,\tilde{n}} \leq s^*_{i,\tilde{n}}(s_j)$, if $y^*_{i,\tilde{n}}(s^*_i(s_j),s_j) = 1$, then Firm i prefers $s^*_i(s_j)$ instead of s_i as a response to s'_j. In other words, under the condition that the best response to the opponent's strategy leads to produce in period $\hat{n}$, Firm i will not decrease its price in $\hat{n}$ if the opponent increases its price in such period.*

Given the pure pricing strategies $s_j \in S_j$ and $s'_j \in \delta_{\tilde{n}}(s_j)$, and $s_i \in S_i$ such that $s_{i,\tilde{n}} \geq s^*_{i,\tilde{n}}(s'_j)$, if $y^*_{i,\tilde{n}}(s^*_i(s'_j),s'_j) = 1$, then Firm i prefers $s^*_i(s'_j)$ instead of s_i as a response to s_j. In other words, under the condition that the best response to the opponent's strategy leads to produce in period $\hat{n}$, Firm i will not increase its price in $\hat{n}$ if the opponent decreases its price in such period, then .

Proof. We provide the proof for the first part of the theorem (we can derive the second part in a similar fashion). Firm i will prefer $s^*_i(s_j)$ instead of any $s_i \in S_i$ as a response to s'_j, if condition $B_i(s^*_i(s_j), s'_j) \geq B_i(s_i, s'_j)$ holds. We can re-write the previous expression in terms of the incomes and costs related to s_j.

$$\begin{aligned} B_i(s^*_i(s_j), s_j) + \gamma_{i,\tilde{n}} \cdot \Delta(s'_j, s_j) \cdot (s^*_{i,\tilde{n}}(s^-_j) - s_{i,\tilde{n}}) \geq \\ B_i(s_i, s_j) + \tau(s'_j, s^*_i(s_j), s_j) - \tau(s'_j, s_i, s_j). \end{aligned} \tag{3.19}$$

From Definition 3.1 , we certainly know that $B_i(s^*_i(s_j), s_j) \geq B_i(s_i, s_j), \forall s_i \in S_i$. Hence, the following expression is a sufficient condition for satisfying expression (3.19).

$$\gamma_{i,\tilde{n}} \cdot \Delta(s'_j, s_j) \cdot (s^*_i(s_j) - s_{i,\tilde{n}}) \geq \tau(s'_j, s^*_i(s_j), s_j) - \tau(s'_j, s_i, s_j). \tag{3.20}$$

Moreover, Lemma 3.3 establishes that $\overline{\tau}(s'_j, s^*_i(s_j), s_j) \geq \tau(s'_j, s^*_i(s_j), s_j)$ and $\underline{\tau}(s'_j, s_j) \leq \tau(s_i(s_j), s'_j, s_j)$, hence the right hand side of the previous expression is upper bounded by $\overline{\tau}(s'_j, s^*_i(s_j), s_j) - \underline{\tau}(s'_j, s_j)$. Thus, if the following expression holds, condition (3.20) is satisfied.

$$\gamma_{i,\tilde{n}} \cdot \Delta(s'_j, s_j) \cdot (s^*_{i,\tilde{n}}(s_j) - s_{i,\tilde{n}}) \geq \overline{h}_i(s^*_i(s_j), s_j) \cdot \gamma_{i,\tilde{n}} \cdot \Delta(s'_j, s_j). \tag{3.21}$$

By simplifying expression 3.21, we obtain $s^*_{i,\tilde{n}}(s_j) - s_{i,\tilde{n}} \geq \overline{h}_i(s^*_i(s_j), s_j)$. The right hand side of this expression is less than 0 for any price larger than $s^*_i(s_j)$, and consequently, the condition will not hold. Nevertheless, we are interested in prices smaller than or equal to $s^*_i(s_j)$, in such cases the condition will hold only if $\overline{h}_i(s^*_i(s^-_j), s^-_j) = 0$, which occurs when $y^*_{i,\tilde{n}}(s_i, s_j) = 1$. □

Theorem 3.3 provides upper ad lower bounds for the price that a firm will set in each period in order to respond to the opponent's pure pricing strategy. We can utilize these bounds for fixing the values of some of the variables w_i of model (3.15) - (3.17), so we can reduce the time needed to find a best response strategy.

3.5 Obtaining a Nash Equilibrium

In this section we describe how to utilize the characterizations introduced in Section 3.4 in order to reduce the time for constructing the payoff matrices of the best responses of the firms. We study three approaches for constructing such matrices: Best Response (Approach *BR*), Ex-ante (Approach *EA*) and Full Characterization (Approach *FC*). Here, we explain the implementation of each approach:

- Approach *BR* constructs pay-off matrices through solving the best response problem for each opponent's pure pricing strategy. We use model (3.7) for computing best response strategies.

- Approach *EA* is similar to Approach *BR*, but we implement the conditions introduced in Corollaries 3.1 - 3.3. So, we decrease the number of responses that should be analyzed for each pure strategy of the opponent. Further, we use Corollaries 3.1 - 3.3 for identifying pure pricing strategies that the opponent will not choose as best responses. In this way, we reduce the number of opponent's strategies to be revised when constructing payoff matrices.

- Approach *FC* uses the same conditions of Approach *EA*, but it takes advantage of the fact that we identify iteratively the best responses to the opponent's strategies. If we implement the bounds derived in Section 3.4.2, we may avoid solving some best response problems.

Implementing the proposed approaches can help to reduce the computational time required to compute an equilibrium between firms in comparison to the computation of the full pay-off matrices. The magnitude of such reductions, however, will depend on how we process the available information. In particular, the computational time associated to Approach *FC* depends on the sequence in which the opponent's strategies are analyzed. For example, let us consider three pure pricing strategies of the opponent for $N = 3$: $s_1 = (L_j, L_j, L_j)$, $s_2 = (M_j, L_j, L_j)$, $s_3 = (U_j, L_j, L_j)$. If we solve the best response problems by following the sequence s_1, s_2, s_3, once we obtain $s_i^*(s_1)$ we may establish lower bounds for $s_i^*(s_2)$ and $s_i^*(s_3)$. After obtaining $s_i^*(s_2)$, the accuracy of the lower bound for $s_i^*(s_3)$ may be improved. But the sequence s_1, s_3, s_2 could be more efficient, because once we obtain $s_i^*(s_1)$ and $s_i^*(s_3)$, there may be a lower bound and an upper bound for $s_i^*(s_2)$. In this way, solving the problem related to $s_i^*(s_2)$ may not be required for computing an equilibrium between firms. Based on this example, we propose an algorithm for obtaining the best responses of a firm. In Figure 3.1a we show the dynamic of this algorithm for $N = 2$. Each cell represents a pure pricing strategy of the opponent, and the number in the cells symbolizes the sequence in which the opponent's pure pricing strategies are analyzed. The dashed line represents the potential lower and upper bounds that we derive from solving the best response problem to a strategy of the opponent. Further, Figure 3.1b shows the dynamics of a procedure consisting in solving the best response problem by following an ordinal sequence of the strategies of the opponent. Clearly, the opportunity of getting upper and lower bounds for the best response problems is increased by implementing the proposed sequence.

3.6 Numerical Experiments

In this section we investigate the efficiency in obtaining a NE (or MNE) when implementing the approaches introduced in Section 3.5. Moreover, we aim to get insights about the following aspect of the LSP under competition:

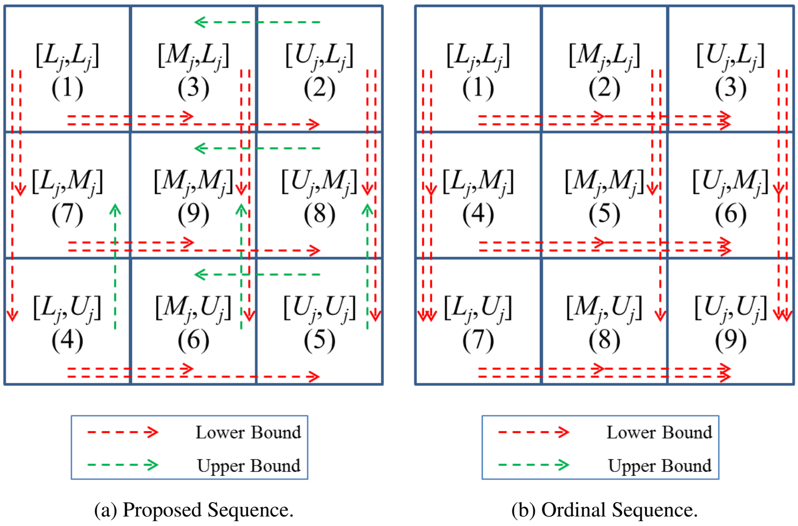


Figure 3.1: *Sequence for exploring pure pricing strategies of the opponent.*

- The sensitivity of the computational time when varying the different parameters of the problem.
- The changes of the price strategy chosen by the firms when varying the different parameters of the problem.
- The effect of the demand seasonality on the pricing strategy of the firms.
- The effect of implementing a constant price policy on the profits of the firms. Our analysis provides evidence about the potential interest of dynamic pricing.
- For operations with capacity constraints, we analyse the effects of the production capacity of the firms on the profits.

In our analysis we assume identical firms, i.e., we test instances in which the production activities, the prices and the demand functions are characterized by identical parameters for both firms. For each instance, we derive the NE that maximizes the sum of the profit of the firms, in other words, we maximize $\mathbf{max}(\overline{b}_i + \overline{b}_j)$ subject to the constraints (3.8) - (3.13). If such equilibrium does not exist we assume the firms randomize their price strategies, i.e., we compute a MNE for such scenarios. Furthermore, we assume that the prices from L_i, M_i and U_i $\forall i = 1,2$ already include the unit production cost of the items from the final price, thus we set the value of such cost to 0.

Our tests were implemented in the Java language. The computer used was a 6-Core Intel Xeon 2×2.66 GHz with 48 GB of RAM. MIP problems are solved using Gurobi 4.6.1 (http://www.gurobi.com/).

3.6.1 Computational time

We compare the computational time for solving the pricing problem when we utilize Approach *BR*, Approach *EA* and Approach *FC*. We analyze 1,792 instances, which are the result of the full factorial combination of the following parameters: $N = 2,...,7$; $f_i = \{2.00, 8.00\}$; $\beta_{i,n} = \{0.25, 2.00\}$; $\gamma_{i,n} = \{0.20 \cdot \beta_{i,n}, 0.80 \cdot \beta_{i,n}\}$ (it is natural to assume that $\gamma_{i,n} \leq \beta_{i,n}$ since in general the demand for a product is more sensitive to its own price); $h_i = \{0.20, 0.80\}$; $L_i = \{1.00, 4.00\}$ and $M_i - L_i = U_i - M_i = \{0.40, 1.60\}$. Further, we utilize the parameter $\underline{d} = \{0.40, 1.60\}$ to define the minimum demand in each period, where $\alpha_{i,n} = \underline{d} + \beta_{i,n} \cdot U_i - \gamma_{i,n} \cdot L_i$. This experiment does not consider the effect of seasonality.

Table 3.1 summarizes the average computational time for calculating payoff matrices through the approaches considered. For Approach *EA* and Approach *FC*, we compute the percentage reduction of the number of best response problems that have to be solved in comparison to Approach *BR*. Further, we include the time for computing the NE based on constraints (3.8) - (3.13).

	Approach					
	BR	EA		FC		NE
N	Time	Time	% red.	Time	% red.	Time
2	0.1	0.0	64.2%	0.0	76.0%	0.0
3	0.2	0.1	69.3%	0.0	85.0%	0.1
4	1.0	0.5	72.4%	0.1	89.4%	0.3
5	4.4	2.2	74.4%	0.3	91.7%	1.3
6	15.8	7.8	75.6%	0.9	93.7%	6.0
7	62.5	27.4	76.5%	3.2	94.1%	24.4

Table 3.1: *Average Computational Time for Obtaining a Nash Equilibriul by Different Approaches (in seconds).*

From Table 3.1 we get insights about the relative efficiency of the different approaches. As we could expect, the computational time associated to Approach *BR* increases exponentially with the number of periods. This time is significantly reduced when implementing Approach *FC*, because this approach computes the payoff matrices of the firms by solving the best response problem for a part of the whole number of the opponent's pure pricing strategies.

Given that the computational time of the approaches considered grows exponentially with N, we modify our analysis for larger instances. Now, we measure the number of opponent's strategies revised in 1 minute by the different approaches. Based on such measure, we estimate the time for computing the set of best responses of the firms. Note that, this indicator may underestimate the performance of Approach *FC*, because this approach is more efficient in reducing calculations when the number of strategies increases (the links between pure pricing strategies of Figure 3.1b increase with N). For this new experiment we analyze instances where $f_i = 2.00$, $\beta_{i,n} = 2.00$, $\gamma_{i,n} = \{0.20 \cdot \beta_{i,n}, 0.5 \cdot \beta_{i,n}, 0.80 \cdot \beta_{i,n}\}$, $h_i = 0.80$, $L_i = 1.00$, $M_i - L_i = U_i - M_i = 0.40$ and $\underline{d} = 0.40$. Figure 3.2 summarizes the results.

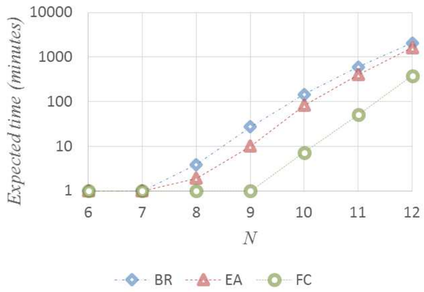
For large instances, FC Approach computes the sets of best responses in a time shorter than the other approaches. For instance, Approach *BR* requires around 100 minutes for obtaining the set of best responses when $N = 9$, while Approach *FC* requires around 1 minute for the same task.

Once the set of best responses is computed, we should obtain a NE between such sets. The last column of Table 3.1 shows that the computational time for obtaining a NE increases with the number of periods. This trend is explained by two facts: (i) the set of best response strategies increases with N and (ii) the LSP that we should solve for obtaining the payoff between best response strategies becomes more complex for larger values of N. Nevertheless, the computational time for obtaining a NE is in general shorter than the time for identifying the set of best responses.

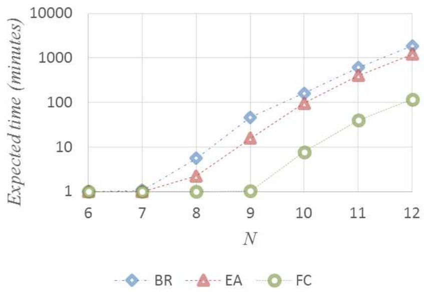
In order to identify the characteristics of the firms that lead to larger computational times, we analyze the sensitivity of the computational time with respect to changes in the different parameters of the problem. With this end, we use two indicators: (i) TBR/TFC, the ratio between the computational time for determining the sets of best responses of the firms when implementing Approach *BR* and Approach *FC* and (ii) TBR/TNE, the ratio between the computational time for the sets of best responses when using Approach *BR* and the time for calculating a NE. Large values of TBR/TFC indicate that implementing Approach *FC* leads to significant reductions of the computational time. Large values of TBR/TNE implies that determining the sets of best responses corresponds to the bottleneck for solving the pricing problem, therefore, any improvement in this step will reduce the total time for setting prices.

We vary indistinctly the value of the different parameters of the problem based on the following starting instance: $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, and $\Delta p = 1$. Figure 3.3 summarizes the values of TBR/TFC and TBR/TNE.

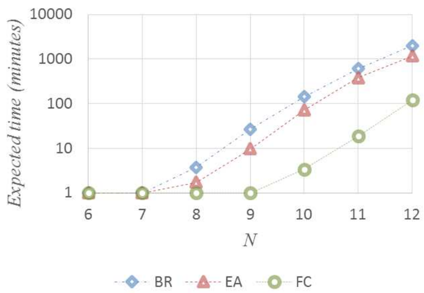
In general, Approach FC reduces by a factor of at least 10 the computational time for determining the set of best responses in comparison to Approach *BR*. The convenience of utilizing Approach FC is reinforced by the fact that in most of the instances considered the ratio TBR/TNE is large. Nevertheless, there are two instances that constitute exceptions. First, TBR/TNE is close to 1 when $\Delta p = 1.2$ as we observe in Figure 3.3f, Even though the time for computing a NE is similar to the time of getting the set of best responses, in this instance the two computational times are negligible. Second, when $\gamma/\beta \geq 0.7$ in Figure 3.3d, computing the set of best responses does not correspond to the bottleneck of the whole pricing problem. Moreover, the com-



(a) $\frac{\gamma_{i,n}}{\beta_{i,n}} = 0.2$



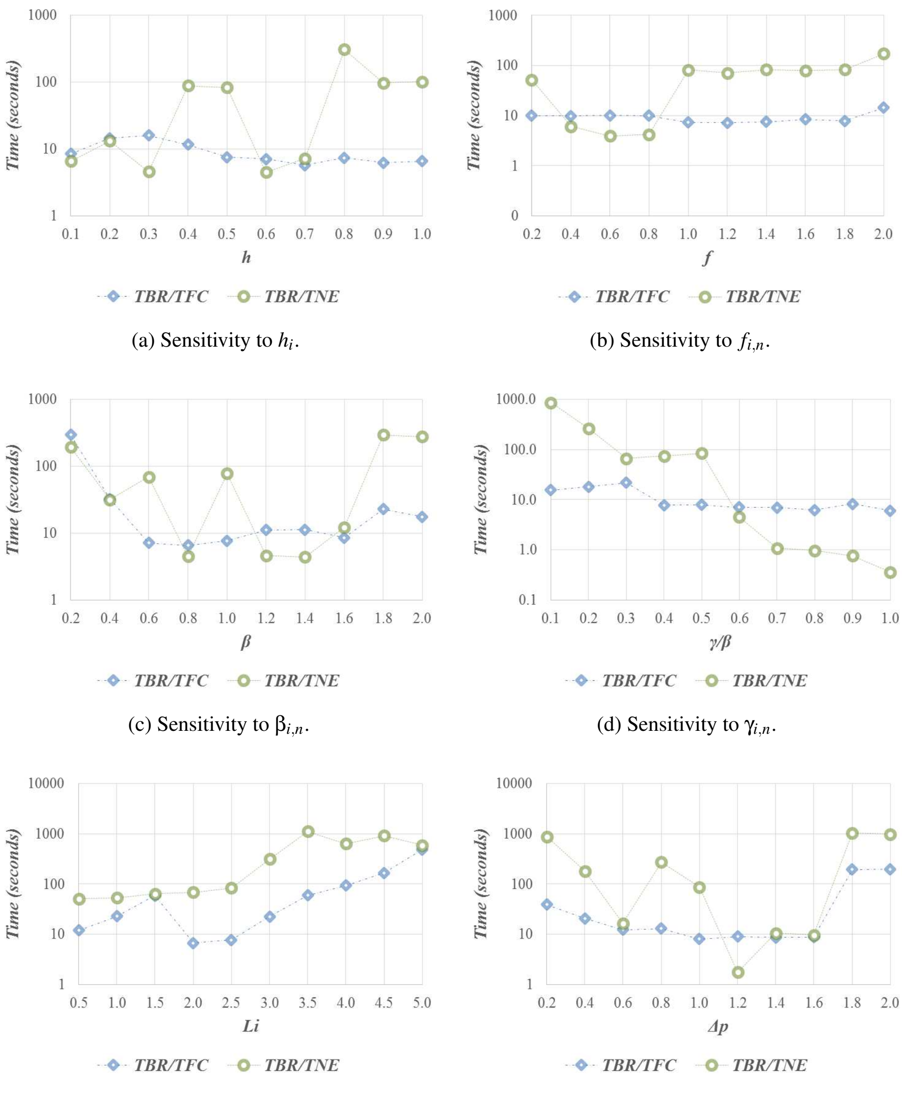
(b) $\frac{\gamma_{i,n}}{\beta_{i,n}} = 0.5$



(c) $\frac{\gamma_{i,n}}{\beta_{i,n}} = 0.8$

Figure 3.2: *Estimated Computational Time for Computing Payoff Matrices by Using Different Approaches.* $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$.

putational time for obtaining a NE is significant for this instance, and consequently, Approach FC helps to reduce the computational time, but its global impact is less significant.



(a) Sensitivity to h_i.

(b) Sensitivity to $f_{i,n}$.

(c) Sensitivity to $\beta_{i,n}$.

(d) Sensitivity to $\gamma_{i,n}$.

(e) Sensitivity to L_i.

(f) Sensitivity to Δp.

Figure 3.3: *Computational Time for Solving the Pricing Problem of the Firms by Implementing Different Approaches.* $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$.

3.6.2 Price Selection

We conduct experiments to measure how the price that each firm selects for a period varies with the parameters of the problem. For these experiments we generate instances from the base set of parameters utilized on the sensitivity analysis of the computational time. Tables 3.2 - 3.7 summarize the results for these experiments.

As we observe from Table 3.2, the total demand accepted by a firm during the production horizon tends to be stable with respect to the changes in the holding costs. Nevertheless, the distribution of the prices during the horizon depends on the value of h_i. For instance, when the holding cost is low, a firm tends to manufacture during the first periods, while it keeps the demand in the last periods low (higher prices). On the contrary, when the holding cost is higher, the production is more frequent, and the peaks of demands (low prices) are more disperse in the horizon.

	Period					
h	1	2	3	4	5	6
0.1	1.5	1.5	2.5	2.5	2.5	2.5
0.3	1.5	2.5	2.5	1.5	2.5	2.5
0.5	1.5	2.5	2.5	1.5	2.5	2.5
0.7	1.5	2.5	1.5	2.5	1.5	2.5
0.9	1.5	2.5	1.5	2.5	1.5	2.5

Table 3.2: *Price Selection for Different Values of* h_i. $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $L_i = 2.5$, *and* $\Delta p = 1$.

Table 3.3 shows the relation between price strategies and fixed production costs. The NE leads to higher prices when the value of the set-up cost is more significant. This phenomenon is explained by two facts. First, when the set-up cost is high, a firm tends to produce in less periods in order to reduce the number of periods in which the production is carried out, so this firm has an incentive to increase its prices. Second, when the set-up cost is high, a firm should increases prices in order to cover such costs.

	Period					
f	1	2	3	4	5	6
0.2	1.5	1.5	1.5	1.5	1.5	1.5
0.6	1.5	2.5	1.5	2.5	1.5	2.5
1.0	1.5	2.5	2.5	1.5	2.5	2.5
1.4	1.5	2.5	2.5	1.5	2.5	2.5
1.8	1.5	2.5	2.5	1.5	2.5	2.5

Table 3.3: *Price Selection for Different Values of* f_i. $T = 6$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$

As we could expect, the firms will choose higher prices when the demand for a product is less sensitive to its own price (smaller values of $\beta_{i,n}$), as we show in Table 3.4.

β	Period 1	2	3	4	5	6
0.25	3.5	3.5	3.5	3.5	3.5	3.5
0.75	2.5	2.5	2.5	2.5	2.5	2.5
1.25	1.5	2.5	2.5	1.5	2.5	2.5
1.75	1.5	2.5	1.5	2.5	1.5	2.5
2.25	1.5	1.5	1.5	1.5	1.5	1.5

Table 3.4: *Price Selection for Different Values of* $\beta_{i,n}$. $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$

In terms of the level of substitution between products, larger values of the ratio $\frac{\gamma_{i,n}}{\beta_{i,n}}$ implies that the demand in each period is less sensitive to the price between firms. Therefore, the price that firms select for each period is higher when $\frac{\gamma_{i,n}}{\beta_{i,n}}$ increases.

$\frac{\gamma}{\beta}$	Period 1	2	3	4	5	6
0.1	1.5	1.5	2.5	1.5	1.5	2.5
0.3	1.5	2.5	2.5	1.5	2.5	2.5
0.5	1.5	2.5	2.5	1.5	2.5	2.5
0.7	2.5	2.5	2.5	2.5	2.5	2.5
0.9	2.5	2.5	2.5	2.5	2.5	2.5

Table 3.5: *Price Selection for Different Values of* $\frac{\gamma_{i,n}}{\beta_{i,n}}$. $T = 6$, $f = 5$, $\underline{d} = 0.5$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$.

Now, we focus on the changes of the prices of the firms. Tables 3.6 - 3.7 show the relation between the prices selected by the firms and changes of the values of prices and price differences, respectively. We observe that the firms tend to choose high prices when the part of demands that is independent of the prices is larger (see Table 3.6) or when the difference between prices is high (see Tables 3.7). Note that, when the minimum price is large ($L_i = 3.5$ or $L_i = 4.5$ in Table 3.6), the price may not increase anymore. This is because less price sensitive demand makes the price decision more independent between periods. Moreover, the income function is concave, then the discrete price selected by the firms tends to be close to the optimal continuous price. The effect is less significant when the NE provides the minimum sum of profits, given that the firms will not necessarily maximize the mentioned convex function.

			Periods					
L_i	M_i	U_i	1	2	3	4	5	6
0.5	1.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
1.5	2.5	3.5	2.5	2.5	3.5	2.5	2.5	3.5
2.5	3.5	4.5	2.5	3.5	3.5	2.5	3.5	3.5
3.5	4.5	5.5	3.5	3.5	3.5	3.5	3.5	3.5
4.5	5.5	6.5	4.5	4.5	4.5	4.5	4.5	4.5

Table 3.6: *Price Selection for Different Values of L_i. $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, and $\Delta p = 1$.*

			Periods					
L_i	M_i	U_i	1	2	3	4	5	6
1.5	1.7	1.9	1.5	1.5	1.5	1.5	1.5	1.5
1.5	2.1	2.7	1.5	1.5	2.7	1.5	1.5	2.7
1.5	2.5	3.5	1.5	3.5	3.5	1.5	3.5	3.5
1.5	2.9	4.3	4.3	4.3	4.3	4.3	4.3	4.3
1.5	3.3	5.1	5.1	5.1	5.1	5.1	5.1	5.1

Table 3.7: *Price Selection for Different Values of Δp. $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, and $h_i = 0.5$.*

3.6.3 Impact of Demand Seasonality

We study the changes of the pricing strategies of the firms when seasonality of the demand arises. In order to model seasonality, we assume that the demand sensitivity to prices is constant, but the amount of demand that is independent of the prices will vary with the time, i.e. the value of $\alpha_{i,n}$ depends on n. We study two scenarios of seasonality: increasing demand ($\alpha_{i,n} \leq \alpha_{i,n+1}$) and decreasing demand ($\alpha_{i,n} \geq \alpha_{i,n+1}$). We set the value of $\alpha_{i,n}$ based on the parameter m, such that $\alpha_{i,n} = \alpha_{i,1} + n \cdot m$ for increasing demands and $\alpha_{i,n} = \alpha_{i,N} + (N-n) \cdot m$ for decreasing demands. The rest of the parameters corresponds to the base set considered in the sensitivity analysis of the computational time, but we vary the level of price sensitivity. We summarize the results in Table 3.8.

β	m	Period 1	2	3	4	5	6
	-20%	3.5	3.5	3.5	3.5	3.5	3.5
	-10%	3.5	3.5	3.5	2.5	3.5	3.5
0.2	0%	2.5	2.5	3.5	2.5	3.5	3.5
	10%	2.5	3.5	3.5	3.5	3.5	3.5
	20%	2.5	3.5	3.5	3.5	3.5	3.5
	-20%	2.5	2.5	2.5	2.5	2.5	2.5
	-10%	2.5	2.5	2.5	1.5	2.5	2.5
0.5	0%	1.5	2.5	2.5	1.5	2.5	2.5
	10%	1.5	2.5	2.5	2.5	2.5	2.5
	20%	1.5	2.5	2.5	2.5	2.5	2.5
	-20%	2.5	2.5	1.5	2.5	1.5	2.5
	-10%	1.5	2.5	1.5	2.5	1.5	2.5
0.8	0%	1.5	1.5	1.5	2.5	1.5	2.5
	10%	1.5	2.5	1.5	2.5	1.5	2.5
	20%	1.5	2.5	1.5	2.5	1.5	2.5

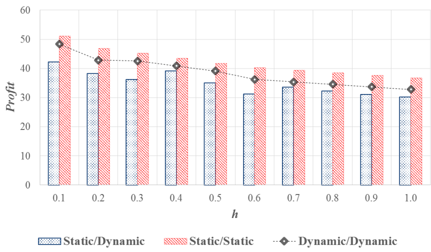
Table 3.8: *Price per Period for Different Demand Seasonality.* $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$.

Decreasing demands ($m < 0$) imply that the part of the demand that is not sensitive to the prices ($\alpha_{i,n}$) is relative larger than the sensitive part during the first periods, so a firm will tend to choose higher prices in these periods, as we can observe in Table 3.8. When the demand is increasing with the time ($m > 0$), a firm will set higher prices in comparison to $m = 0$, but the increases will not necessarily occur in the last periods in which the demand is less sensitive to the prices.

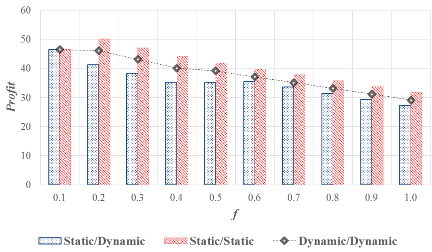
3.6.4 Impact of Dynamic Pricing

If each firm sets an unique price for the a production horizon, then the complexity of the problem may decreased, because the set of space of pure pricing strategies is reduced. We measure how firms can gain or loose by setting static prices in comparison to the dynamic pricing. With this in mind, we compare the profit of a firm for three scenarios: both firms set dynamic prices, both firms set static prices, a firm set static prices, but its opponent set dynamic prices. In our experiments we utilize the same instances considered in Section 3.6.2 The results are summarized in Figure 3.4.

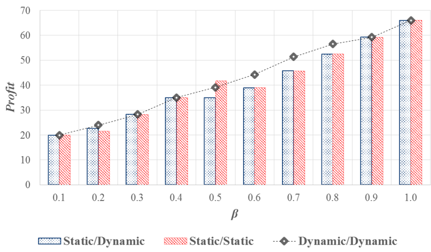
In general the firms can be hurt by setting static pricing in comparison to the dynamic pricing. Even so, under certain scenarios playing simultaneously static prices could be more convenient for the firms (see Figures 3.4d , 3.4b and 3.4d). The firms



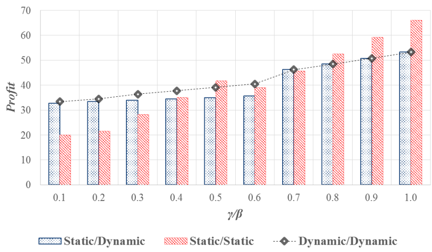
(a) Sensitivity to h_i.



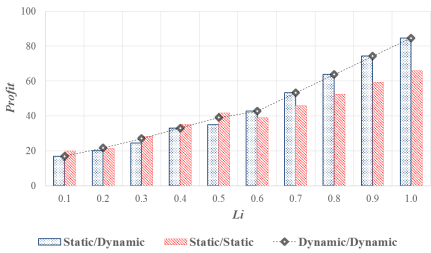
(b) Sensitivity to $f_{i,n}$.



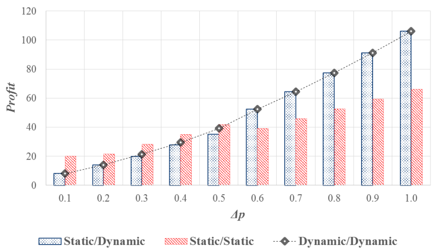
(c) Sensitivity to $\beta_{i,n}$.



(d) Sensitivity to $\gamma_{i,n}$.



(e) Sensitivity to L_i.



(f) Sensitivity to Δp.

Figure 3.4: *Profit of a Firm by Static Pricing and Dynamic Pricing.* $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, and $\Delta p = 1$

may utilize this fact as an incentive to choose simultaneously static pricing strategies, but these simultaneous decisions may not be feasible in practice, because they constitute a collusive practice between competitors. Therefore, our analysis focuses on the effect of setting unilaterally static prices on the profit of a firm. As we show in Figures 3.4a , 3.4b and 3.4d, the firms cannot get advantages from such decisions, i.e. static pricing may reduce the profit of a firm in comparison to dynamic pricing.

3.6.5 Capacitated Lot Sizing Problem

We analyze the effect of increasing the production capacities of both firms on the sum of the profits of the firms. Given that larger production capacities lead to reductions of the production costs of the firms, we could expect that such capacity increase leads to larger profits. Our numerical experiments, however, provide evidence that the sum of the profit of the firms may decrease when the capacity increases.

By setting additional capacity, the firms may benefit from the economies of scales arising on their production activities, so the firms can accept larger demands by setting lower prices. This is favorable for the company if the incomes from the demand increase surpass the reduction of prices. An example of this phenomenon is $\gamma_{i,n}/\beta_{i,n} = 0.2$ in Figure 3.5. Nevertheless, if both firms reduce simultaneously their prices, the increase of demand will be less significant because the substitution effect of the products, therefore, the profit of the firms may be reduced. For instance, when $\gamma_{i,n}/\beta_{i,n} = 0.8$ the firms can gain from increasing their capacities from 5.5 to 6.5, but they will be hurt if they continue increasing their capacities.

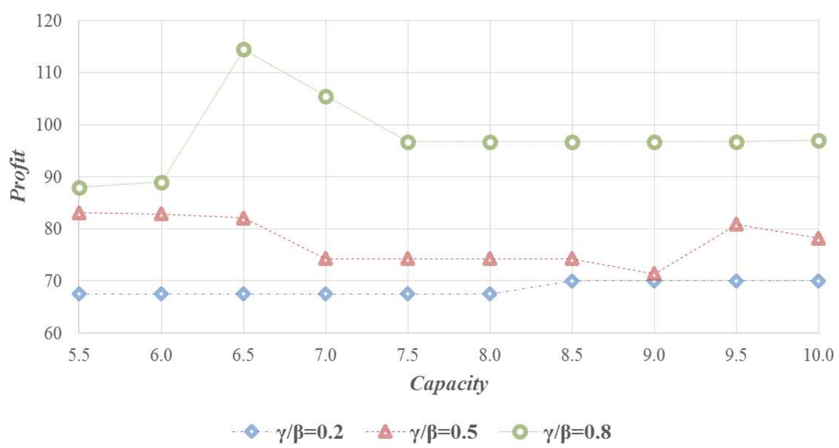


Figure 3.5: *Sum of the Profits of the Firms for Different level of Production Capacities.* $T = 6$, $f = 5$, $\underline{d} = 0.5$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$

Another interesting discussion is how a firm modifies its profits by increasing unilaterally its production capacity. Table 3.9 summarizes the result of these experiments. Table 3.9 shows that increasing unilaterally the production capacity is not always convenient for a firm.

Cap	$\frac{\gamma}{\beta} = 0.2$	$\frac{\gamma}{\beta} = 0.5$	$\frac{\gamma}{\beta} = 0.8$
5.5	33.8	41.6	44.0
6.0	34.1	38.5	47.5
6.5	35.0	42.9	48.4
7.0	35.0	43.4	48.4
7.5	35.0	43.4	48.4
8.0	35.0	41.8	48.4
8.5	35.0	41.8	48.4
9.0	35.6	41.8	48.4
9.5	35.6	41.8	48.4
10.0	35.6	44.2	48.5

Table 3.9: *Profit of a Firm by Increasing Unillaterally its Capacity.* $T = 6$, $f = 5$, $\underline{d} = 0.5$, $\beta_{i,n} = 1.25$, $\gamma_{i,n} = 0.5 \cdot \beta_{i,n}$, $h_i = 0.5$, $L_i = 2.5$, *and* $\Delta p = 1$

3.7 Summary and Conclusions

We study the problem of pricing and production of two firms that offer mutually substitute products. For the pricing decisions, we assume that each firm maximize its profits by choosing a price from a discrete set of options. Related to the production decisions, we model the operations of each firm as a LSP. Computing an equilibrium between the pricing strategies requires the full mapping of the firms pay-off, so this calculation may be computational demanding when the size of the problem increases. We propose a methodology for identifying the pricing strategies that will lead the equilibrium between firms, in such a way that we can discard the computation of the pay-off for other strategies.

Our methodology consists of two steps. In the first step, we characterize the pricing strategies of a firm that constitute the set of best responses to the opponent's strategies. Given an opponent's strategy, we calculate the best response by solving a MIP formulation. In the second step, we calculate a NE between the strategies between the strategies arising from the first step. Again, we calculate the NE by solving a MIP formulation. Through numerical experiments we show that our methodology lead to significant reduction of time for solving the pricing problem of the firms in comparison to a full mapping of the pay-off of the firms.

Although our methodology considers two competitors and three discrete prices, to extend it to more competitors or more prices is straightforward. Indeed, when more firms compete or the set of available prices is larger we could expect that our methodology will lead to larger reductions of the computational time, because our characterization will significantly shrink the space of strategies. Further, although we assume a linear relation between demand and prices, the way we build our methodology can be applied to other type of demand functions.

We also provide managerial insights that can enlighten the actions of decision makers. We show that two firms that compete between each other can obtain larger profits by making dynamic pricing decisions in comparison to setting static pricing strategies. Furthermore, we show that when the operations of the firms have capacity limitations, if the firms increase their capacity, they may be hurt for that action. Moreover, this counter intuitive phenomenon may also occur when a firm increases its capacity unilaterally.

We propose to extend our work to the following research avenues:

- To examine the trade-off between the price and the delivery time that firms promise to customers when working under competition. In practice, the demand of customer depends on the price, but it can also depend on other factors such as the promised delivery time. In that scenario, promising a shorter time will bring more demand, however, this will require additional efforts when managing operations.

- Obtaining a NE when the firms have incomplete information about the opponent's strategies. The procedure for obtaining such NE have a similar structure that the problem we study, however, the complexity in reaching the equilibrium will increase due to the uncertainty. We glimpse two options for the formulation of the operations planning under uncertainty: (i) to model the operations of each firm as as a Stochastic LSP, (ii) to plan the operations in order to obtain a plan for operations that is robust with respect to the changes of the opponent's strategy (the formulation of the problem is a Robust LSP).

Chapter 4

Planning Operations for Revenue Maximization Under Lead Time Constraints

4.1 Introduction

This article is motivated by the case of a cast iron manufacturer. This company is specialized in the production of cast iron pieces for industrial equipments. Most of its orders are either for the preventive maintenance of installations or for the building of new facilities. Such projects are scheduled with long lead times, but it is very important that the pieces be delivered on time because the plant where the pieces have to be installed will have to be (at least partially) stopped for the maintenance or installation activities to take place and obviously the duration of such stoppage should be minimized. A different type of orders received by the company corresponds to corrective maintenance when a breakdown occurs in a plant, in those cases a new cast iron piece is needed to restart the facility. Given that production at the customer is stopped because of the breakdown, a much faster service is required but the company can charge higher prices for such "emergency" orders. Moreover, the bargaining power of the customer is much weaker in such circumstances. Given its finite production capacity, the company cannot always accept all orders and should sometimes forego a regular order in order to keep some possibility to accept an urgent order later on. This dilemma is faced by many suppliers confronted with urgent requests that are potentially very profitable but could be very disruptive if not taken into account in their planning. Typical examples from the service sector include heating ventilation air-conditioning (HVAC) companies. The installation of new systems is typically a large project with a relatively long lead time. In contrast, when a system fails it could block the operations of a customer that is then willing to pay a higher fee for speedy action. In a very different context, suppliers of the fashion industry are known to combine orders from large vendors and more profitable orders for high fashion clothes which often require fast delivery (see e.g. deB Harris and Pinder, 1995, Barut and Sridharan, 2005). The question facing the supplier is how much capacity should be set aside for the urgent high margin demand, given the inherent unpredictability of this type of orders.

To address this question, we build a model with a supplier that handles two demand classes, that we will refer to as regular and urgent respectively. The regular orders are typically characterized by longer processing times, longer lead times but lower margins, while the urgent orders have shorter processing times, shorter lead-times and higher margins. If the supplier accepts orders without foresight it is likely that at some point, when an urgent order arrives, the supplier will be unable to accept this order as her short term capacity is already entirely committed for regular orders (that were booked earlier with a longer lead time). Given the difference in margin between the two classes, this situation causes some loss of revenue. On the other hand, rejecting a regular order in anticipation for potential urgent orders that do not materialize, also causes some revenue loss.

This tradeoff has clear similarities with other revenue management problems. The distinct feature is that when an order is accepted the supplier keeps some flexibility. For example, if a regular order necessitates 10 days of work and the lead time is 20 days, in most cases the customer does not care about the days during which the order is effectively produced as long as it is finished on time. In revenue management terms, if we consider that the capacity available during each period is a distinct *product*, an order requiring more than one period of work is in fact reserving several *products*. But the supplier has some flexibility in assigning the *products* to the order and does not need to make a commitment at the time of reservation. Such consideration is reasonable and represent the reality at many firms in which jobs can be interrupted and resumed in later periods. One can draw a parallelism with the network revenue management problem but where the supplier can accept a reservation without committing to specific legs in the network, the only commitment is on the origin and destination points in the network.

Our contribution are summarized as follows:

- We derive a Markov Decision Process (MDP) formulation of the order acceptance problem of two customer classes with lead time constraints. Likewise for the network revenue management problem, the size of this formulation quickly excludes the possibility of solving it exactly for larger size instances. We develop a family of approximate formulations parametrizable to range from a coarse approximation to the original full formulation. This makes it possible to choose between speed of solution and precision of result. What is of particular interest is that for each formulation, we can compute an upper and lower bound on the exact result. This last feature is rather uncommon for revenue management problems and is particularly interesting to make sure the adequate level of approximation is chosen in the proposed family of formulations.

- Through a numerical study we show that the proposed heuristics allow to obtain near-optimal solutions in a tractable time. We also show how the potential benefit of revenue management is commensurate with operational flexibility. In our setting operational flexibility consists in the slack between the promised lead time for an order class and the processing time needed for such order.

The remainder of the Chapter is structured as follows. In Section 4.2 we discuss related work. In Section 4.3 we provide a detailed description of our problem. In Section 4.4 we introduce an MDP formulation and discuss its resolution to obtain the optimal admission policy. In Section 4.5 we propose two heuristic formulations of the problem based on different levels of state aggregation and report on a numerical study of our proposed formulations in Section 4.6. Section 4.7 investigates the impact of operational flexibility on the benefit of revenue management. Finally, Section 4.8 summarizes the main conclusions and identifies future research directions.

4.2 Literature Review

Our work belongs to the growing literature of Perishable Asset Revenue Management (PARM) which deals with the problem of allocation of scarce resources to different demand classes. Talluri and van Ryzin (2004) give a comprehensive overview of this topic. The first applications were for the airline industry by Littlewood (1972), and extended by Belobaba (1987), Wollmer (1992), and Brumelle and McGill (1993). In addition to airlines, typical service applications are in hotel management and car rental (Kimes, 1989, Bertsimas and Sim, 2003, Talluri and van Ryzin, 2004, Bitran and Mondschein, 1995, Geraghty and Johnson, 1997). Gradually, new applications appeared for very different environments such as: MTO manufacturing (Balakrishnan et al., 1996, Barut and Sridharan, 2005, Spengler et al., 2007), project management (Herbots et al., 2007, 2010) and health care (Gupta and Wang, 2008, Dobson et al., 2011).

A stream of literature related to lead-times decisions focuses on due-date quotation and scheduling problems in order to allocate the available capacity to incoming orders (see e.g. Kaminsky and Hochbaum, 2004, Keskinocak and Tayur, 2004, for an extensive literature review). Most of these works assign dynamically lead times to incoming orders depending on the state of the system and sequencing policies (see e.g. Duenyas, 1995, Duenyas and Hopp, 1995, Kapuscinski and Tayur, 2007). Kapuscinski and Tayur (2007) propose a dynamic programming approach to address the problem of lead-time quotation for two demand classes when customers are not equally sensitive to waiting. Lead time quotation is used to ensure that the capacity is allocated in such a way that all demands can be delivered on time. So, firms can change the quoted lead time based on the system state. Motivated by the prevalence of static lead time policies (see e.g. Cheng and Gupta, 1989, Hopp and Sturgis, 2000, Keskinocak and Tayur, 2004), we consider a different problem in which lead times are constant and exogenously given. Since the capacity may not be enough to cater all demands, the decision becomes accepting or rejecting orders depending on the system state.

Gupta and Wang (2007) consider an order acceptance problem in which the lead time requirement for regular orders is modelled as a soft operational constraint. It is assumed that tardiness cost is incurred if regular orders are not filled within their lead time window while urgent orders must be filled in the current period once accepted. The authors propose a multi-dimensional MDP whose optimal solution turns out to be a threshold based policy. This solution property is a consequence of the simplicity of their model setting, which leads to a well-structured value function. In contrast, our model is more general, assuming some flexibility in catering urgent orders – the lead time for urgent order does not necessarily need to be one. The state space in our problem is defined in a different way as in Gupta and Wang (2007), because their representation involves tracking the backlogging information for every demand class and becomes particularly inefficient when there are multiple demand classes, which greatly limits its application. Our representation ”encodes” in itself the capacity allocation decisions and therefore is more efficient.

The following references focus on acceptance decision problems where the lead times must be strictly respected, as in our case. Germs and Van Foreest (2011) study an order acceptance problem with multiple customer classes with a common deadline, setup times and scheduling constraints. The problem is modelled as a Markov chain controlled by a threshold policy. The authors provide a numerical study for small instances which are computationally tractable. In contrast to their work, we provide efficient alternative methods to treat the state space explosion. Barut and Sridharan (2005) study an order acceptance problem involving multiple demand classes that differ in terms of price, lead time and demand pattern. The authors propose a nested rationing policy which fulfils incoming orders as much as possible while preserving a certain level of capacity for more profitable future orders. The proposed policy is computed using a myopic heuristic method, that does not take the evolution of the capacity into account. Consequently, the efficiency of the heuristic is hurt by the simplified estimation of the future available capacity. Our formulation keeps track more accurately of the capacity evolution for this type of problem, leading to highly efficient policies.

Literature related to development and implementation of different approximate methods to reduce the computational time required to solve large instances of dynamic problems is abundant. Examples of methods developed to solve MDPs include state aggregation techniques (see e.g. Mendelssohn, 1982, Bean et al., 1987, Aldhaheri and Khalil, 1991, Hu and Wu, 2000, Zhang and Baras, 2001, Van Roy, 2006, Jia, 2011), embedding/time aggregation approaches via value and policy iteration (see e.g. Cao et al., 2002, Leizarowitz and Shwartz, 2008, Sun et al., 2007, Arruda and Fragoso, 2011, Cheng and Zhang, 2012) and convergence acceleration methods for value iteration (Almudevar and Arruda, 2012). In this research, we apply state aggregation techniques for a problem of order acceptance when customer classes differ in their lead times. There is, however, a difference between the previous works and our research: they are designed for solving MDP in general forms, while our aggregation method is motivated by the specific characteristics of the problem of interest. In this sense, our work is in the same spirit with Xu et al. (2007). The authors model a ticket queue problem with a Markov chain and used state aggregation technique to reduce the size of the state space based on the structure of the ticket queue. In this research, we implement state aggregation techniques to aggregate some information of the states of the system within the lead time window of the incoming orders. Thereby, the infinite horizon problem is solved, and the state space of the problem is reduced.

There are some similarities with the network revenue management problem where the dimension of the MDP models quickly make it impossible to solve exactly even small size problems. Consequently, research in this area has concentrated on different approximation techniques, notable examples of this stream of investigation include Bitran and Mondschein (1995), Talluri and van Ryzin (1998), Bertsimas and Sim (2003), Adelman (2007), Kunnumkal and Topaloglu (2008), Zhang and Adelman (2009), Zhang (2011). The flexibility aspect present in the problem studied here calls for different types of approximations.

4.3 Model Description

We consider the order acceptance problem for a firm serving two customer classes with different profit margins and lead times. The time horizon is infinite and consists of discrete periods. In each period, the firm is subject to a limited processing capacity, which is normalized to 1.

We consider that all uncertainty about the processing time is known when the order is placed. This assumption is motivated by two reasons: on the one hand, in practice most uncertainty is resolved during the ordering process; on the other hand, if the remaining uncertainty is too large it is not possible to promise due dates without either large safety lead times or a low utilization. In the cases that motivated our work, we observed that the remaining processing time uncertainty after the order is placed is dealt with using some type of recourse action such as overtime or renegotiation of the due-date. These recourse actions play only a secondary role in the management of capacity and are beyond the scope of our investigation.

The two classes of demand will be denominated, urgent and regular orders (indexed by $k = \{1, 2\}$, respectively). The demand (in terms of processing time) for class k at time t will be denoted D_{kt}. We suppose the random variables take integer values and are iid between time periods and independent between classes. If $D_{kt} = 0$ there is no demand for class k in period t. The profit margin per unit capacity of a class k customer is r_k and its lead time is L_k. Urgent orders are more lucrative but come with a short lead time, while regular orders are not as profitable, yet have a looser lead time. Accordingly, we assume $r_1 > r_2 > 0$ and $L_1 < L_2$. Figure 4.1 shows the structure of the problem. We suppose that only a single order of each class can arrive during any period t, this means that the demand D_{kt} cannot be partially accepted, the firm either accepts the order and hence commits to deliver the order before its due date or declines the order and gets no revenue. On-time delivery positively influences customer experience and reinforces the long term image of firms. When confirming orders, firms usually allocate sufficient capacity for processing the orders in a timely manner. Since the processing capacity is deterministic in our model, it is reasonable to plan with no tardiness.

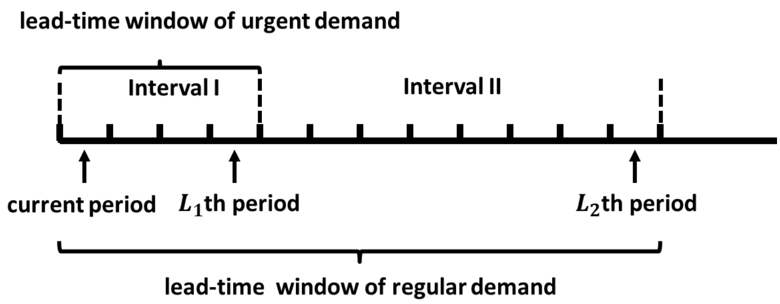


Figure 4.1: *Partition of Lead Time Window of Regular Demand.*

Consequently, the main decision faced by the firm is how many orders to accept in order to maximize its long-run net profit. However, we restrict our attention to the order acceptance decisions of regular orders. We assume that regular orders can be selectively rejected while urgent orders can only be passively rejected as a result of insufficient available capacity. This is because urgent orders often arise in emergency circumstances, and thus are granted with priority over regular orders.[1]

In each period, the sequence of events is as follows. A regular order, if any, arrives first, and the firm decides whether to accept it or not. Then, an urgent order, if any, arrives, it will be accepted as long as there is enough available capacity. Finally, the firm uses the capacity in the current period for processing the order with the earliest due date. In fact, the sequence of arrivals does not matter; one can assume that the urgent order arrives before the regular order, or both arrive simultaneously, without complicating the model.

In the remainder of the Chapter we will assume a somewhat simpler structure for

[1]The modeling framework in our research is general enough that the option of rejecting urgent orders can be easily incorporated without substantial modification of the model.

the demand. We assume

$$D_{kt} = \begin{cases} 0, & \text{with probability } 1 - p_k, \\ B_k, & \text{with probability } p_k, \end{cases}$$

and thus the inter-arrival times are geometrically distributed in each demand class. This simplifies the notations in the following sections and our numerical tests indicate that the distribution of the demand has no qualitative impact on our results.

4.4 MDP Formulation for the Optimal Steady-State Policy

The crux of developing the MDP formulation of this problem is to find an efficient representation of the usage of future capacity. Our representation builds on the idea that when an order is accepted, some future capacity will be "reserved". Reservation refers to *provisionally* allocating the available capacity of the latest periods to process an order without incurring tardiness. This gives the maximum flexibility to accept the future orders. This notion is also used in the model formulation of Kapuscinski and Tayur (2007). Note that, the allocation is provisional because the firm may start processing earlier the order if there is no order to deliver before this one. When an order is (partially) processed before its provisionally allocated time slot, some capacity is freed to process future orders. For example, when accepting an order with a lead time of 5 and a size of 2, the capacity of 4th and 5th periods from the current period shall be reserved (provided that they are available for reservation), if the order will be processed as soon as possible in the earliest-due-date (EDD) sequence.

To reserve capacity for new orders, the firm needs to calculate the total available capacity for reservation within the lead time windows of regular and urgent orders. Since the lead time window of urgent orders is contained within that of the regular orders, it suffices to calculate the total available capacity within L_1 periods, and that between (L_1+1)th and L_2th periods, i.e. Interval I and Interval II of Figure 4.1, respectively. However, this aggregate information cannot fully characterize the evolution of the system. Note that the (L_1+1)th period, (i.e., the first period of Interval II), will be shifted by one period and thus become the L_1th period in the next period, (i.e., the last period of Interval I). Without the information regarding how the reserved capacity is distributed in Interval II, it is impossible to know whether the shifted capacity is reserved or not in order to update the available capacity in Interval I and II in the next period. Therefore, it is necessary to keep track of the distributional information in Interval II, but only of aggregate information in Interval I.

We now introduce the notation to be used in our formulation.

- $\mathbf{x}$: is the reservation vector, it keeps track of capacity that has been reserved for processing; $\mathbf{x}[0] \in \{0,1,\ldots,L_1\}$ denotes the total reserved capacity until the L_1th period, (i.e., in Interval I). For $j = 1,2,\ldots,L_2 - L_1$, $\mathbf{x}[j] = 1$ if the capacity of $(L_1 + j)$th period is reserved, and $\mathbf{x}[j] = 0$ otherwise. Note that in a reservation vector we do not distinguish whether the capacity is reserved for urgent orders or regular orders.

- $\mathbf{y}$: is the cumulative (available capacity) vector[2] of $\mathbf{x}$; for $j = 0,1,\ldots,L_2 - L_1$, $\mathbf{y}[j]$ denotes the *total* available capacity until the $(L_1 + j)$th period, i.e., $\mathbf{y}[j] = L_1 + j - \sum_{i=0}^{j} \mathbf{x}[i]$. For a given cumulative vector $\mathbf{y}$, its corresponding reservation vector $\mathbf{x}$ can be calculated as follows: $\mathbf{x}[0] = L_1 - \mathbf{y}[0]$, and for $j = 1,2,\ldots,L_2 - L_1$, $\mathbf{x}[j] = \mathbf{y}[j-1] - \mathbf{y}[j] + 1$.

- a: represents the admission decision for regular orders; $a = 1$ if the firm "admits" the regular order, and $a = 0$ otherwise.

- $\mathbf{D} \equiv (D_1, D_2)$: is the demand vector in each period.

- $R(\mathbf{x},\mathbf{D},a)$: denotes the profit generated from $\mathbf{D}$ for a given admission decision a, if the reservation vector at the beginning of the current period is $\mathbf{x}$.

We define the system state as the reservation vector $\mathbf{x}$ at the beginning of a period, before the arrival of regular and urgent orders. It is easy to check that $\mathbf{x}[L_2 - L_1] = 0$ for any system state, because the last element of the system state cannot be reserved by orders that arrived in earlier periods. Thus, the system state space essentially involves $L_2 - L_1$ variables and its size is $(L_1 + 1) \cdot 2^{L_2 - L_1 - 1}$. Let $\widetilde{\mathbf{x}}$ be the reservation vector updated from $\mathbf{x}$ after accepting/rejecting D_2. Additionally, let $\widehat{\mathbf{x}}$ be the system state in the next period, this is the reservation vector updated from $\widetilde{\mathbf{x}}$ after accepting/rejecting D_1 and processing, if any, is carried out in the current period. Further, let $\widetilde{\mathbf{y}}$ and $\widehat{\mathbf{y}}$ be the corresponding cumulative vectors of $\widetilde{\mathbf{x}}$ and $\widehat{\mathbf{x}}$, respectively. Thus, $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are functions of $\mathbf{x}$, $\mathbf{D}$ and a: $\widehat{\mathbf{x}} = \widehat{\mathbf{x}}(\mathbf{x},\mathbf{D},a)$, $\widehat{\mathbf{y}} = \widehat{\mathbf{y}}(\mathbf{x},\mathbf{D},a)$. According to the average reward criteria (Ross, 1983, Bertsekas, 1995, Putermann, 2005), we provide the dynamic programming formulation as follows,

$$V(\mathbf{x}) + g = \mathbb{E}_{\mathbf{D}} \left\{ \max_a \left\{ R(\mathbf{x},\mathbf{D},a) + V\left(\widehat{\mathbf{x}}(\mathbf{x},\mathbf{D},a)\right) \right\} \right\}, \forall \mathbf{x}, \tag{4.1}$$

[2] $\mathbf{y}$ is introduced to facilitate the notations related to transitions between system states, which are composed of reservation vectors. There is a one-to-one correspondence between a reservation vector and its cumulative vector.

together with the dynamics described by equations (4.2)-(4.7),

$$\widetilde{\mathbf{y}}[j] = \begin{cases} \mathbf{y}[j] - (D_2 - (\mathbf{y}[L_2 - L_1] - \mathbf{y}[j]))^+, & \text{if } a = 1 \text{ and } \mathbf{y}[L_2 - L_1] \geq D_2, \\ \mathbf{y}[j], & \text{otherwise,} \end{cases} \tag{4.2}$$

$$\text{for } j = 0, 1, , \ldots, L_2 - L_1,$$

$$\widehat{\mathbf{y}}[j] = \begin{cases} \min\{\widetilde{\mathbf{y}}[j+1] - D_1, L_1 + j\}, & \text{if } \widetilde{\mathbf{y}}[0] \geq D_1, \\ \min\{\widetilde{\mathbf{y}}[j+1], L_1 + j\}, & \text{otherwise,} \end{cases} \tag{4.3}$$

$$\text{for } j = 1, 2, \ldots, L_2 - L_1 - 1,$$

$$\widehat{\mathbf{y}}[L_2 - L_1] = \widehat{\mathbf{y}}[L_2 - L_1 - 1] + 1, \tag{4.4}$$

$$R(\mathbf{x}, \mathbf{D}, a) = R_1(\widetilde{\mathbf{x}}, D_1) + R_2(\mathbf{x}, D_2, a), \tag{4.5}$$

$$R_2(\mathbf{x}, D_2, a) = \begin{cases} r_2 \cdot D_2, & \text{if } a = 1 \text{ and } \mathbf{y}[L_2 - L_1] \geq D_2, \\ 0, & \text{otherwise,} \end{cases} \tag{4.6}$$

$$R_1(\widetilde{\mathbf{x}}, D_1) = \begin{cases} r_1 \cdot D_1, & \text{if } \widetilde{\mathbf{y}}[0] \geq D_1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

In the solution of Equation (4.1), g represents the steady state expected profit per period. Equations (4.2)-(4.4) characterize the transition between system states $\mathbf{x}$ and $\widehat{\mathbf{x}}$, with the help of their corresponding cumulative vectors $\mathbf{y}$ and $\widehat{\mathbf{y}}$. Specifically, Equation (4.2) describes how $\mathbf{y}$ is updated to $\widetilde{\mathbf{y}}$, and Equations (4.3)-(4.4) further describe how $\widetilde{\mathbf{y}}$ is updated to $\widehat{\mathbf{y}}$. When updating the system state, all accepted orders are scheduled in the reservation vector as late as possible within their lead time windows to allow for maximal flexibility to process the orders. Equations (4.5)-(4.7) calculate the profit generated during transitions. Specifically, $R_1(\widetilde{\mathbf{x}}, D_1)$ represents the profit from urgent orders and $R_2(\mathbf{x}, D_2, a)$ represents the profit from regular orders.

Example 1. *Consider an instance with the following parameters: $L_1 = 4$, $L_2 = 8$, $B_1 = 2$, $B_2 = 3$. Suppose the reservation vector at the beginning of the current period is $\mathbf{x} = [2, 0, 1, 1, 0]$, meaning there are 2 units of capacity reserved in Interval I and 2 units reserved in Interval II, totaling 4 units of available capacity for fulfilling regular orders. If a regular order arrives ($D_2 = B_2 = 3$), and the decision is to accept it, 1 unit of available capacity in Interval I and 2 units in Interval II will be reserved for processing the order, and the reservation vector will be updated to $\widetilde{\mathbf{x}} = [3, 1, 1, 1, 1]$ or $\widetilde{\mathbf{y}} = [1, 1, 1, 1, 1]$ (equation (4.2)). However, this leaves with only 1 unit of available capacity within the lead time window of urgent orders, ($i.e., \widetilde{\mathbf{y}}[0] = 1$) and therefore, there is not enough room to accommodate any urgent order (since $B_1 = 2$). Finally, the capacity of the current period is used to process one unit of order in Interval I, and one unit of reserved capacity is shifted from Interval II into Interval I. Therefore, the reservation vector observed at the beginning of next period becomes $\widehat{\mathbf{x}} = [3, 1, 1, 1, 0]$ or $\widehat{\mathbf{y}} = [1, 1, 1, 1, 2]$ (equations (4.3) and (4.4)).*

4.5 State Reduction Heuristics

The formulation described in Section 4.4 leads to an optimal steady-state policy for accepting/rejecting regular orders. However, the size of the formulation, i.e., the number of states, increases exponentially in $L_2 - L_1$. For example, if $L_1 = 5$ and $L_2 = 25$, the resulting formulation has 3,145,728 states. Thus, we seek to develop more compact heuristic formulations.

The complexity of the formulation is largely related to keeping track of the distributional information in Interval II. To reduce the complexity, one plausible idea is to somehow aggregate the distributional information in Interval II, so that the MDP formulation can be reduced to involve fewer states. The reduced formulation can then be used to generate heuristic policies.

We start with the Full Aggregation Heuristic (FAH) that completely ignores the distributional information in Interval II. Though being extremely compact, the FAH does not always lead to near-optimal solutions. Consequently, we propose the Partial Aggregation Heuristic (PAH) which keeps the most "important" distributional information intact while aggregating the rest in Interval II. A major advantage of this approach is that one can easily control the tradeoff between computation efforts and optimality.

4.5.1 Full Aggregation Heuristic

We propose a new MDP formulation based on *aggregate reservation vectors* as opposed to reservation vectors in the original formulation. For a reservation vector $\mathbf{x}$, we define its corresponding aggregate reservation vector as $\mathbf{x}_f = (\mathbf{x}_f[0], \mathbf{x}_f[1])$, in which $\mathbf{x}_f[0] = \mathbf{x}[0]$ and $\mathbf{x}_f[1] = \sum_{j=1}^{L_2-L_1} \mathbf{x}[j]$, i.e., $\mathbf{x}_f[0]$ corresponds to the total reserved capacity in Interval I, and $\mathbf{x}_f[1]$ corresponds to that in Interval II. Note that there can be multiple reservation vectors mapping to the same aggregate reservation vector. Further, we define $\mathbf{y}_f = (\mathbf{y}_f[0], \mathbf{y}_f[1])$ as the *aggregate cumulative vector*, in which $\mathbf{y}_f[0] = L_1 - \mathbf{x}_f[0]$ and $\mathbf{y}_f[1] = L_2 - \mathbf{x}_f[0] - \mathbf{x}_f[1]$, i.e., $\mathbf{y}_f[0]$ (respectively $\mathbf{y}_f[1]$) corresponds to the total available capacity in the lead time window of urgent (respectively regular) demands. The one-to-one mapping between an aggregate reservation vector and its aggregate cumulative vector still holds.

The new system state is defined as the aggregate reservation vector in the beginning of a period, and therefore only involves two dimensions. One way to think of the new system state is that each one groups multiple system states in the original formulation into a "super state" (Figure 4.2), resulting in a significantly shrunk state space. In other words, the size of the aggregated state space is reduced from $(L_1+1) \cdot 2^{(L_2-L_1-1)}$ to $(L_1+1) \cdot (L_2-L_1)$.

Then we characterize how one new system state transits to another for a given admission policy a and demand pattern D under the new state space. Note that transitions

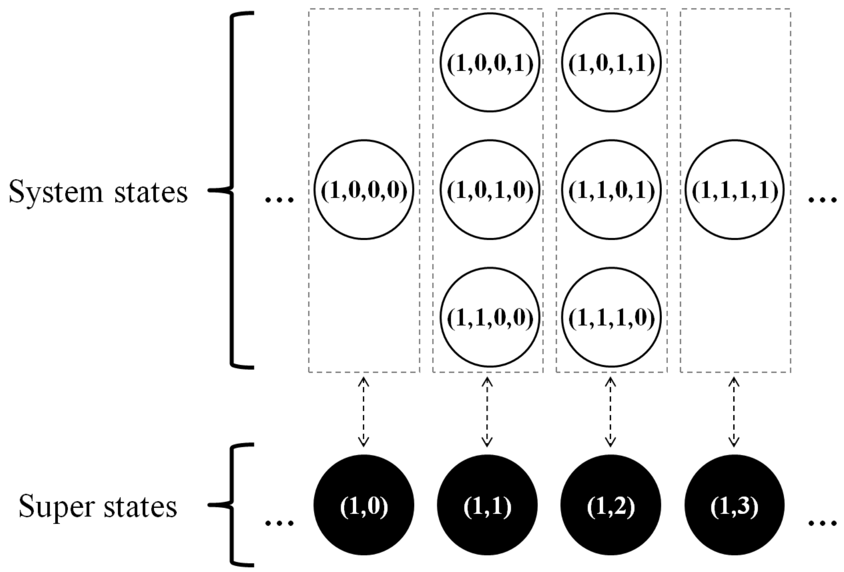


Figure 4.2: *The New System States Defined by "Super States". This illustrative example shows the states whose first element is 1 for an instance in which $L_2 - L_1 = 4$.*

triggered by accepting/rejecting demands and processing can be easily characterized, but there is one step of transition that cannot be properly defined due to the aggregation, that is, how the aggregated reservation vector at the end of one period evolves to the one at the beginning of the next period. As discussed previously, without distributional information in Interval II, it is impossible to decide whether the shifted unit of capacity is reserved or not. To address this issue, we heuristically account for the transfer of capacity based on assumptions regarding how the reserved capacity is distributed in Interval II. We consider three scenarios as follows.

- **Optimistic scenario.** Assuming that all reserved capacity in Interval II (excluding the last unit) are distributed *as late as possible*, the shifted unit of capacity is reserved only if Interval II (excluding the last unit) is "full", i.e., all the capacity in this area has been reserved. "Optimistic" refers to the fact that the assumption results in an overestimation of the available capacity in Interval I.

- **Pessimistic scenario.** Assuming that all reserved capacity in Interval II (excluding the last unit) are distributed *as early as possible*, the shifted unit of capacity is available only if Interval II (excluding the last unit) is "empty", i.e., none of the capacity in this area has been reserved. "Pessimistic" refers to the fact that the assumption results in an underestimation of the available capacity in Interval I.

- **Realistic scenario.** Assuming that all reserved capacity in Interval II (excluding the last unit) are distributed *uniformly*, the shifted unit of capacity is reserved with a probability, defined as the proportion of reserved capacity in Interval II (excluding the last unit).

Among the three scenarios, we regard the realistic scenario to be closest to the reality. Note that the distribution in Interval II is influenced by two forces: reservation and updating from one period to the next. Reservation fills the capacity from the end to the front, while updating from one period to the next shifts the capacity forward by one period. Hence, we know that the real distribution must be less extreme than the ones in the optimistic and pessimistic scenarios. Nonetheless, th scenario leading to a better policy is yet to be examined.

These assumptions do not deal with the last unit of capacity in Interval II, because whether it is reserved or not can be explicitly characterized without introducing additional dimensions: it is always available at the beginning of one period, and it will become reserved whenever some regular demand is accepted in the current period.

Let $\widetilde{\mathbf{x}}_f$, $\widehat{\mathbf{x}}_f$, $\widetilde{\mathbf{y}}_f$ and $\widehat{\mathbf{y}}_f$ be the aggregate version of vectors $\widetilde{\mathbf{x}}$, $\widehat{\mathbf{x}}$, $\widetilde{\mathbf{y}}$ and $\widehat{\mathbf{y}}$, respectively. In addition, we define $\overline{\mathbf{y}}_f$ as the aggregate cumulative vector after accepting/rejecting urgent orders but before processing is carried out, i.e. $\overline{\mathbf{y}}_f$ serves as an intermediary between $\tilde{\mathbf{y}}_f$ and $\widehat{\mathbf{y}}_f$. We characterize the transition between $\mathbf{x}_f$ and $\widehat{\mathbf{x}}_f$ as follows.
Accepting/rejecting a regular order:

$$(\widetilde{\mathbf{y}}_f[0], \widetilde{\mathbf{y}}_f[1]) = \begin{cases} (\mathbf{y}_f[0] - (D_2 - (\mathbf{y}_f[1] - \mathbf{y}_f[0]))^+, \mathbf{y}_f[1] - D_2), & \text{if } a = 1 \text{ and } \mathbf{y}_f[1] \geq D_2, \\ (\mathbf{y}_f[0], \mathbf{y}_f[1]), & \text{otherwise.} \end{cases} \tag{4.8}$$

Accepting/rejecting an urgent order:

$$(\overline{\mathbf{y}}_f[0], \overline{\mathbf{y}}_f[1]) = \begin{cases} (\widetilde{\mathbf{y}}_f[0] - D_1, \widetilde{\mathbf{y}}_f[1] - D_1), & \text{if } \widetilde{\mathbf{y}}_f[0] \geq D_1, \\ (\widetilde{\mathbf{y}}_f[0], \widetilde{\mathbf{y}}_f[1]), & \text{otherwise.} \end{cases} \tag{4.9}$$

Processing and updating to the next period:

$$(\widehat{\mathbf{y}}_f[0], \widehat{\mathbf{y}}_f[1]) = \begin{cases} (\min\{\overline{\mathbf{y}}_f[0], L_1 - 1\}, \min\{\overline{\mathbf{y}}_f[1] + 1, L_2\}), & \text{with probability } \pi \\ (\min\{\overline{\mathbf{y}}_f[0], L_1 - 1\} + 1, \min\{\overline{\mathbf{y}}_f[1] + 1, L_2\}), & \text{with probability } 1 - \pi. \end{cases} \tag{4.10}$$

in which the value of π is contingent on whether the last unit of capacity in Interval II is reserved or not. Some additional notation follows: let ψ be the total available capacity in Interval II after processing, i.e., $\psi = \overline{\mathbf{y}}_f[1] - \min\{\overline{\mathbf{y}}_f[0], L_1 - 1\}$. Next, we discuss the value of π in the different cases.
Case 1: $\widetilde{\mathbf{y}}_f = \mathbf{y}_f$ (the last unit of capacity in Interval II is available)

- In the optimistic scenario, if $\psi = 1$, then $\pi = 1$; otherwise, $\pi = 0$.

- In the pessimistic scenario, if $\psi \neq L_2 - L_1$, then $\pi = 1$; otherwise, $\pi = 0$.
- In the realistic scenario, $\pi = 1 - \frac{\psi-1}{L_2-L_1-1}$.

Case 2: $\widetilde{\mathbf{y}}_f \neq \mathbf{y}_f$ (the last unit of capacity in Interval II is reserved)

- In the optimistic scenario, if $\psi = 0$, then $\pi = 1$; otherwise, $\pi = 0$.
- In the pessimistic scenario, if $\psi \neq L_2 - L_1 - 1$, then $\pi = 1$; otherwise, $\pi = 0$.
- In the realistic scenario, $\pi = 1 - \frac{\psi}{L_2-L_1-1}$.

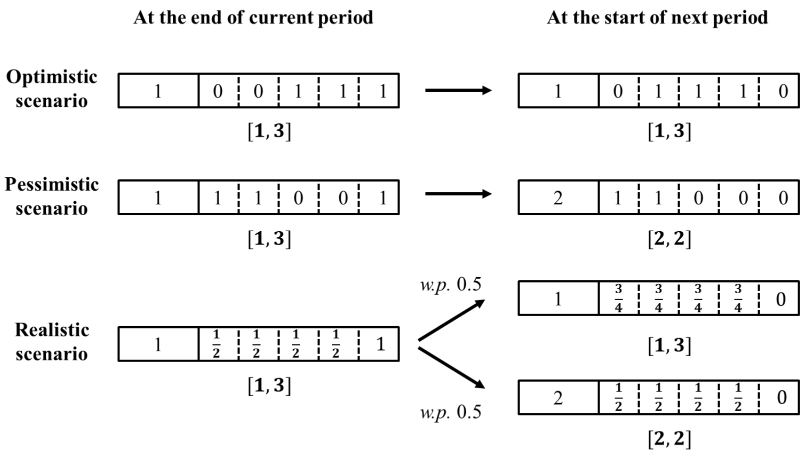


Figure 4.3: *Transition of Aggregated Reservation Vectors Between Two Periods.*

Figure 4.3 shows an example of how an aggregated reservation vector evolves in different scenarios. In this example, $L_1 = 2$ and $L_2 = 7$, the aggregated reservation vector at the end of one period, (i.e., the one after accepting/rejecting demands and processing) is $(1,3)$. Assuming some regular demand is accepted in this period, the last unit in Interval II is reserved. In the optimistic scenario, the shifted unit of capacity is available because the Interval II (excluding the last unit) is not full, and thus the aggregated reservation vector at the start of next period is $\widehat{\mathbf{y}}_f = (1,3)$; in the pessimistic scenario, the shifted unit is reserved, leading to $\widehat{\mathbf{y}}_f = (2,2)$; in realistic scenario, with probability $\pi = 2/4 = 0.5$, the shifted unit is reserved, leading to $\widehat{\mathbf{y}}_f = (2,2)$, and with probability $1 - \pi = 0.5$, the shifted unit is available, leading to $\widehat{\mathbf{y}}_f = (1,3)$.

The profit generated in each aggregated state can be defined in a similar fashion as in (4.5)-(4.7), and a new MDP formulation can be further obtained from optimality equation (4.1) by replacing the system states with their aggregated versions. For simplicity, we omit their representations.

Finally, once an optimal admission policy for the new MDP is obtained, a heuristic policy can be constructed in the following way: for each aggregate state, apply its admission decision to all the corresponding states in the original formulation.

4.5.2 Partial Aggregation Heuristic

The lack of accuracy in characterizing the transitions between aggregate states sometimes leads to significant profit gaps in comparison with the optimal profit, as we will show in the numerical experiments. We wonder whether a more precise characterization of the distributional information in Interval II could improve the performance of the heuristic. We propose a generalized heuristic, the PAH, that partially aggregates the distributional information in Interval II. Specifically, we split Interval II into two parts: (i) Sub-Interval II-A, where the distributional information is precisely tracked, and (ii) Sub-Interval II-B, where the distributional information is fully aggregated. Figure 4.4 shows an example of reservation vectors for different levels of aggregation.

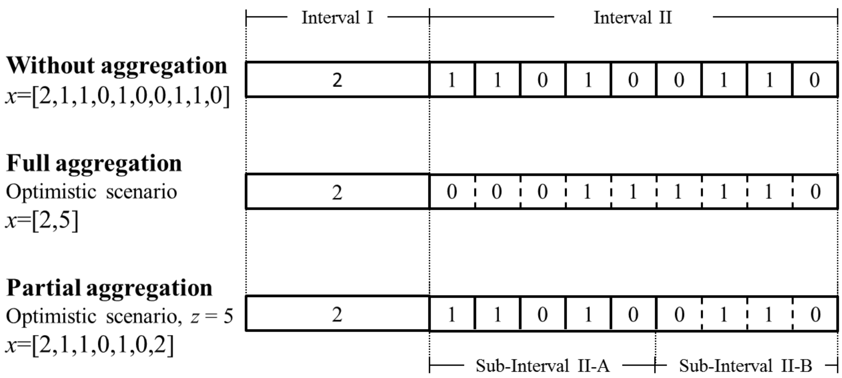


Figure 4.4: *An Example of Reservation Vectors for Different Levels of Aggregation.*

However, we face the same issue as in the full aggregation case, that is, how to account for the transfer of capacity between Sub-Interval II-A and Sub-Interval II-B. Again, we address this issue by heuristic approaches in which the reserved capacity in Sub-Interval II-B is distributed according to the pessimistic, realistic and optimistic scenarios as described in Section 4.5.1. The MDP formulation is derived in the same way; for simplicity, we omit their presentation.

Let $z \in \{0, \ldots, L_2 - L_1 - 1\}$ be the number of periods in Sub-Interval II-A, hereafter referred to as the disaggregation level. This parameter controls the tracking accuracy of the distributional information in Sub-Interval II-A: if $z = 0$, the resulting formulation is actually identical to the full aggregation; if $z \geq L_2 - L_1 - 2$, the resulting formulation coincides with the one without aggregation that provides the optimal steady-state policy. For a given disaggregation level z, the size of the resulting state space is $(L_1+1)\cdot(L_2-L_1-z)\cdot 2^z$. Obviously more disaggregation leads to larger state space. Moreover, we have the following result comparing the optimal profits for different partially aggregated MDP formulations.

Proposition 4.1. *Let $g_o^*(z)$, $g_p^*(z)$ and $g_r^*(z)$ be the optimal expected profit obtained by solving the partial aggregation model with a level of disaggregation z for the optimistic, pessimistic and realistic scenarios, respectively. We have*

$$g_p^*(z) \leq \{g_r^*(z), g^*\} \leq g_o^*(z), \forall z,$$

and $g_o^*(z)$ and $g_p^*(z)$ are non-increasing and non-decreasing with z, respectively.

Proof. The proof is based on two facts: (i) it is always feasible to shorten the lead times of the already accepted orders given that there is available capacity in earlier periods, and (ii) for any aggregate state, the pessimistic scenario implies shorter lead times for already accepted orders than the realistic scenario, which further implies shorter lead times than the optimistic scenario. Therefore, any transition between two aggregate states of scenarios implying shorter lead times can also be achieved with scenarios implying more relaxed lead times. Consequently, for any sample path for the formulation of scenarios implying shorter lead times, we can obtain an identical sample path that provides the same profit with more relaxed scenarios. Thus, the profit for the formulation of the optimistic scenario is at least as large as the profit for the formulation of the realistic scenario, which is further at least as large as the profit for the formulation of the pessimistic scenario. The same logic can be applied for proving the monotonicity property of $g_o^*(z)$ and $g_p^*(z)$, i.e., more aggregation implies shorter (longer) lead times for the pessimistic (optimistic) scenario. □

Proposition 4.1 indicates that $g_p^*(z)$ and $g_o^*(z)$ are the lower and upper bounds for $g_r^*(z)$ and g^*, but $g_r^*(z)$ and g^* are not directly comparable. It also shows that the proposed bounds get tighter as the level of disaggregation z increases.

4.6 Numerical Results

In this section we investigate the actual performance of the policies presented above, more specifically we analyze:

- the computational time required to obtain an acceptance policy with the different formulations.
- the relative profits obtained with the FAH, the PAH, the optimal steady-state policy and two benchmark policies often cited in the literature: the First-Come-First-Served (FCFS) policy and the protection level based (PLB) policy (see Section 4.6.3 for a detailed description of this policy).

- the tightness of the bounds on the profit derived for the FAH and the PAH.

In order to answer those questions, we first compute the heuristic acceptance policies and the optimal policy for the instances small enough to do so. We then simulate the different heuristic policies in order to determine their performance in terms of average gain per period. Indeed, although the gains $g_o^*(z)$ and $g_p^*(z)$ computed for the optimistic and pessimistic policies constitute upper and lower bounds on the optimal gain, the actual gains achieved by those policies – as well as the other policies – cannot be directly derived from the solution of the Markov Decision Process with an aggregate state description.

We simulate the long-run net profit for an acceptance policy by generating demand realizations. The simulation consists of an initial warm-up interval of 100,000 periods. Afterwards, the simulation incorporates an additional 100,000 periods for which the accumulated net profit is recorded. We repeat the process until the simulated net profit converges with a precision of 0.001%. We denote $\hat{g}_f$ and $\hat{g}_m$ the simulated long-run net profit achieved by the FCFS and the PLB policies, and $\hat{g}_o(z)$, $\hat{g}_p(z)$ and $\hat{g}_r(z)$ the simulated long-run net profit achieved by the PAH with a disaggregation level z assuming the optimistic, pessimistic and realistic scenarios, respectively. Note that the gain of the optimal steady-state policy is given directly by the solution of the Markov Decision Process, its value is equal to g^*.

Throughout the experiments, we will use linear programming (LP) to solve all MDP formulations (Ross, 1983, Bertsekas, 1995, Putermann, 2005). Among other tools, we have iteration based methods such as value iteration and policy iteration. They are efficient in finding near-optimal solutions in a short time frame. However, when serving as benchmarks one needs to specify parameters such as number of iterations and optimality gap, which is a nontrivial task. Thus, in order to compare different MDP formulations in terms of their computational time on a fair basis, we choose LP instead.

The acceptance policy simulation routine was implemented in the Java language. The computer used was a 6-Core Intel Xeon 2×2.66 GHz with 48 GB of RAM. LP are solved using Gurobi 4.6.1 (http://www.gurobi.com/).

In the following subsections we introduce certain characteristics of the demand classes that we used to generate instances of the problem (Section 4.6.1). Afterwards, we describe the results of three studies: (1) the computational time for the construction of an acceptance policy for different instances (Section 4.6.2); (2) the profits achieved by the optimal steady-state policy, the FCFS and the PLB policies, and the aggregation heuristics (see Section 4.6.3 and Section 4.6.4 for the FAH and the PAH, respectively); (3) the profits of the FAH and the PAH for large instances where the optimal steady-state policy cannot be computed (Section 4.6.5).

4.6.1 Experiment Settings

The instances we generated are characterized by the following attributes:

Profit structure (ρ). It represents the ratio between the net profits of both demand classes, i.e. $\rho = r_2/r_1$. Without loss of generality, the value of r_1 is normalized to 1, so the value of r_2 is obtained directly from ρ. In our experiment we explore high, moderate and low differences in the profit structure, respectively (see Table 4.1).

Lead-time structure L_1 and L_2. We investigate two aspects of our model that depend on the values of L_1 and L_2. On the one hand, it is clear from the formulation of our model that the computational complexity is closely related to the difference between L_1 and L_2. On the other hand we will see that L_1 influences the performance of the different policies.

Order size structure B_1 and B_2. The sizes of the orders determine the operational flexibility when an order is accepted. We also did some tests with stochastic order sizes, we do not report on this here as the results do not really differ from the deterministic case. The only difference is larger computational times, as a result the computation of the optimal policy is restricted to even smaller instances.

Demand structure (β). It is defined as the ratio between the expected demand rates of both classes, i.e. $\beta = (p_1 \cdot B_1)/(p_2 \cdot B_2)$. We study how this ratio impacts the performance of the proposed formulations. We chose the values of β such that the expected demand of one class is 100% or 50% greater than the other, or the expected demand rates of both classes are equal (see Table 4.1).

Global demand rate (τ). It corresponds to the total expected demand for both classes per period i.e. $\tau = p_1 \cdot B_1 + p_2 \cdot B_2$. We focus on scenarios in which $\tau > 1$, inasmuch as in these scenarios the acceptance decision is most meaningful as some demand will have to be refused. If $\tau < 1$, the decision to not accept a demand is not very relevant.

Profit structure, order size structure and global demand rate are common experimental conditions for main numerical experiments we performed. For these parameters, we perform a full factorial experiment based on the values presented in the first part of Table 4.1. Lead times and order sizes are set depending on the objectives of different numerical experiments performed: analysis of the computational time and the profit comparison of the proposed heuristics with respect to the optimal policy and its bounds. Values of these parameters for each experiment are presented in the second and the third part of Table 4.1, respectively. Note that, p_1 and p_2 are functions of β, τ, B_1 and B_2. Their values will be derived from these parameters.

4.6.2 Computational Times

In this section, we compare the average CPU times needed to calculate the optimal steady-state policy and the heuristic policies. We analyze the FAH and the PAH for

Common Experimental Conditions		Computational Time Experiment		Profit Comparison Experiment	
Attributes	Values	Attributes	Values	Attributes	Values
ρ	$\{0.25, 0.50, 0.75\}$	L_1, B_1	$\{1,3,5,7\}, 1$	L_1, B_1	$3, \{1,3\}$
β	$\{1/2, 2/3, 1, 3/2, 2/1\}$	L_2, B_2	$\{13,15,17,19\}, 1$		$7, \{1,3,5,7\}$
τ	$\{1.2, 1.6, 2.0\}$			L_2, B_2	$\{15,29\}, \{3,5,7,9\}$

Table 4.1: *Setting Values of the Parameters.*

the so-called realistic scenario, which is the most demanding in terms of computational time among the three scenarios considered. Note that, the PAH is tested for different values for the disaggregation level ($z \in \{2,4,6\}$). In addition to the common experimental conditions (see Table 4.1), in this experiment, we fix the order sizes to 1 because this leads to the longest computational times. Similar insights can be obtained for any other combination of values of B_1 and B_2. As already explained, the difference between lead times and their combinations have a direct impact on the size of the state space of the system and, thus, the computational time. To explore this dependency we test a wide range of values of L_1 and L_2 shown in Table 4.1. In total this experiment consists of 342 instances (when discarding the combinations of values of parameters that result in p_1, $p_2 > 1$).

Table 4.2 shows the average CPU times and the size of the system state space for each combination of L_1 and L_2. We observe that the time needed to find the optimal steady-state policy increases very quickly when the difference between L_2 and L_1 increases. In fact, for $L_2 - L_1 \geq 15$ we could not determine the optimal order acceptance policy.

4.6.3 Efficiency of the FAH

We compare the efficiency of the FAH with the optimal steady-state policy, the FCFS and the PLB policies by measuring their relative profits. We also compare the quality of the FAH under each of the three scenarios. In order to compare the efficiencies we compute the optimality gap of an acceptance policy constructed by the heuristic i as follows $Gap_i = (g^* - \hat{g}_i)/g^* \times 100\%$. Note that the FAH is represented by $i = \{o(0), p(0), r(0)\}$ and the FCFS and PLB policy by $i = \{f, m\}$, respectively.

The protection level based (PLB) policy applied in this research is an adaptation of the well-known revenue management approaches that divide the available capacity into two portions: protected (reserved for the high net-profit class) and unprotected (used for both classes). These approaches are commonly applied for the finite-horizon problems. If they are directly implemented over an infinite horizon, the construction of the policy is computationally as intensive as the construction of the optimal steady-state policy. Therefore, in order to reduce the complexity of the existing approaches while at the same time capturing their essence we implement a myopic method which

Table 4.2: *Average CPU Time and Dimension of the State Space of the System for the Optimal Steady-State Policy, the FAH and the PAH Under the Realistic Scenario.*

		Average CPU (seconds)					Size	
				PAH				PAH
L_2	L_1	Optimal	FAH	$z=2$	$z=4$	$z=6$	Optimal	$z=6$
13	1	2.55	0.00	0.01	0.02	0.17	4,096	768
13	3	2.99	0.00	0.01	0.06	0.38	2,048	1,024
13	5	0.26	0.00	0.01	0.06	–	768	768
13	7	0.03	0.00	0.01	0.03	–	256	256
15	1	194	0,00	0.01	0.03	0.30	16,384	1,024
15	3	22	0.00	0.01	0.10	1.02	8,192	1,536
15	5	10	0.00	0.02	0.14	1.20	3,072	1,536
15	7	0.4	0.01	0.02	0.12	–	1,024	1024
17	1	∗	0.00	0.01	0.05	0.51	65,536	1,280
17	3	∗	0.00	0.02	0.16	2.11	32,768	2,048
17	5	105	0.01	0.03	0.28	1.72	12,288	2,304
17	7	8	0.01	0.03	0.29	3.03	4,096	2,048
19	1	∗	0.00	0.01	0.06	0.73	262,144	1,536
19	3	∗	0.00	0.02	0.24	2.45	131,072	2,560
19	5	∗	0.01	0.04	0.44	3.24	49,152	3,072
19	7	1,015	0.01	0.05	0.54	3.52	16,384	3,072

"∗" symbolizes instances for which the optimal steady-state policy cannot be obtained in 1 hour and "–" represents the cases where the PAH is equivalent to the optimal policy because $z \geq L_2 - L_1 - 2$.

determines the amount of protected capacity q that maximizes the expected net profit within L_2. This corresponds to the approximations used in the literature see e.g. Barut and Sridharan (2005). The expected net profit within L_2 is computed with the following expression: $r_1 \cdot E[\min(\bar{D}_1, \max(q, y[L_2] - \bar{D}_2))] + r_2 \cdot E[\min(\bar{D}_2, y[L_2] - q)]$, where the random variable $\bar{D}_i$ represents the amount of demand of class i during L_2 periods. The PLB policy consists in protecting the capacity q that maximizes the previous expression.

In order to evaluate the efficiency of the proposed policies with respect to the optimal steady-state policy, we fix $L_2 = 15$ and $L_1 \in \{3, 7\}$. Under this setting the optimal steady-state policy can be obtained within a tractable time. In order to capture the full essence of different degrees of heterogeneity in demand classes, we consider the different values of order sizes presented in the third part of Table 4.1. The values of other parameters belong to common experimental conditions. In this experiment, our analysis is based on 984 instances (the instances with $p_1, p_2 \geq 1$ are discarded).

Table 4.3 reports on the average optimality gaps for the two values of L_1. The average optimality gap of the FAH is lower than that of the FCFS and the PLB policies. There is also a significant difference between the different implementations of the FAH (realistic, pessimistic and optimistic).

The lowest average optimality gap of the FAH is achieved with the realistic scenario. The strength of the realistic scenario lies in the balance between the excess of protection of capacity for high profitability orders (pessimistic scenario) and the assumption of maximum flexibility for processing incoming orders (optimistic scenario). The results also reveal that the PLB and FAH tend to perform a bit better when L_1 is larger.

Table 4.3: *Average Optimality Gap of the FCFS, the PLB Policy and the FAH Under the Three Scenarios for the Two Values of L_1.*

	FCFS	PLB	FAH		
			Optimistic	Pessimistic	Realistic
$L_1 = 3$	12.92%	7.27%	5.00%	1.90%	0.45%
$L_1 = 7$	13.52%	5.13%	4.98%	0.57%	0.36%
Overall	13.33%	5.81%	4.99%	0.99%	0.39%

We also study the reliability of the different policies when the values of the parameters vary. Figure 4.5 shows the dispersion of the optimality gap of the analyzed methods. The limits of each box represent the first and third quartiles of the measured gaps. The central line corresponds to the mean result. Finally, the bottom and top whiskers represent the fifth and the ninety-fifth percentiles of the optimality gaps. In addition to the advantages achieved in terms of the average optimality gap, the implementation of the FAH with the realistic scenario provides the most reliable performance. Figure 4.6 provides more details about the optimality gap of the analysed instances when utilizing the FAH with the realistic scenario. In particular, the histogram presented in Figure 4.6 shows that the first three categories for which $Gap_r < 1\%$ cover around 90% of the analyzed instances.

The relative advantage of the realistic scenario is born from its accuracy in representing the actual distribution of the reserved capacity in Interval II. Specifically, we expect the assumed distributional scenario leads to a good estimation of the transition of reserved capacity from Interval II to Interval I. With this in mind, we compare the value π for the different distributional scenarios with the observed frequency that such transition occurs. We obtain the observed frequency by simulating the transition between element $x[1]$ and $x[0]$ when implementing the optimal policy. Figure 4.7 provides examples of the observed frequency for different values of B_2 and the assumed values of π when the policy is obtained by the FAH with the realistic scenario. When the value of order size of regular orders is small with respect to its lead time

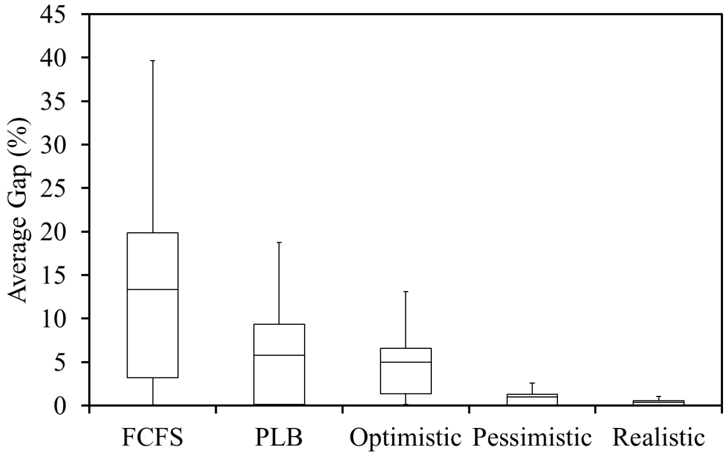


Figure 4.5: *Dispersion of the Optimality Gap of Different Heuristic Methods and Distributional Scenarios for the FAH.*

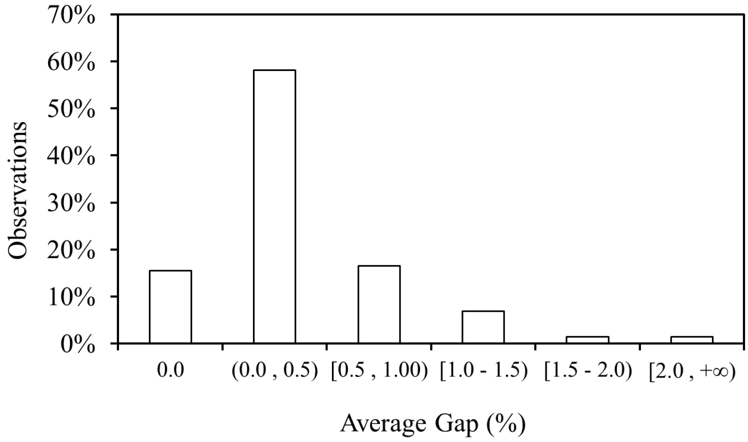


Figure 4.6: *Dispersion of the Optimality Gap of the Realistic Scenarios for the FAH.*

(i.e., $B_2 = 2$ in Figure 4.7), the values of π under the realistic scenario are the closest to the observations. Nevertheless, when the value of B_2 increases (i.e., $B_2 = 4$) the pessimistic scenario seems to be a better assumption for low reserved capacity in Interval II (we remind the reader that under the pessimistic scenario $\pi = 1$ for $x_f[1] > 0$). However, such states are not likely to occur, as we see in Figure 4.7 (probabilities of being in a state for the two values of B_2 are presented by dashed lines with markers). In fact, the realistic scenario estimates π better than the pessimistic scenario for the states where the reserved capacity is equal to 4, which are also the states most likely to occur. Moreover, the sum of the square of the errors of the estimation of the realistic and pessimistic scenario provides more evidence about the advantage of assuming a uniform distribution. In particular, for the case $B_2 = 4$, such sum is 0.019 and 0.013 for the pessimistic and realistic scenarios, respectively.

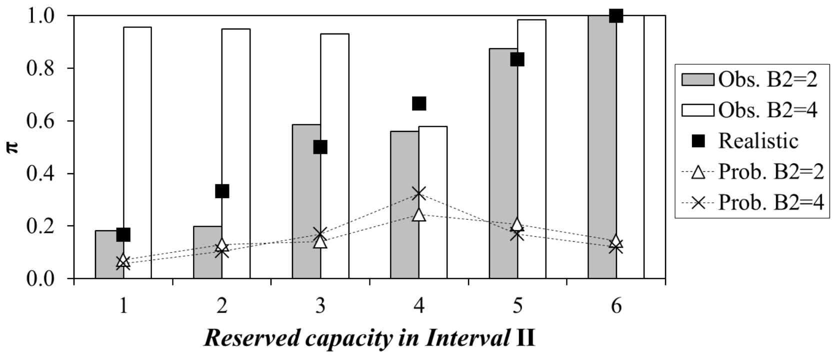


Figure 4.7: *Observed Values and Estimated Values (when the FAH under the Realistic Scenario is Implemented) of* π*. Results for* $L_1 = 4$, $L_2 = 11$, $p_1 = 0.1$, $p_2 = 0.5$, $r_2 = 0.5$ *and* $B_1 = 1$.

Our numerical results allow us to get a glimpse of the structure of the heuristic policy obtained through the FAH with the realistic scenario. This structure corresponds to a threshold in the values of $x_f[0]$ and $x_f[1]$. In other words, if we fix the value of $x_f[1]$ incoming regular orders are accepted up to a certain value of the amount of orders in the first interval. Analogously, if we fix the value of $x_f[0]$ incoming regular orders are accepted if the reserved capacity in the second interval is smaller than or equal to some threshold. Figure 4.8 shows an example of this type of policy.

We analyse the sensitivity of the threshold value obtained through the FAH with the realistic scenario to changes on some parameters of the problem, as we show in Figure 4.9a - Figure 4.9d. We can observe that the threshold values for the first and second intervals decrease with the value of B_2. This is because accepting regular orders with larger order sizes will imply less available capacity for processing the future arrivals of urgent orders. Also, the threshold values have an inverse relation with the values of the probabilities p_1 and p_2, since larger values of such probabilities lead to a reduction of the probability of having idle capacity in the incoming periods, therefore rejecting regular orders could be convenient in order to make some capacity available for urgent orders. The opposite is true for the threshold values and r_2, given that the reduction of the difference between the revenues of the order classes, make regular orders more attractive."

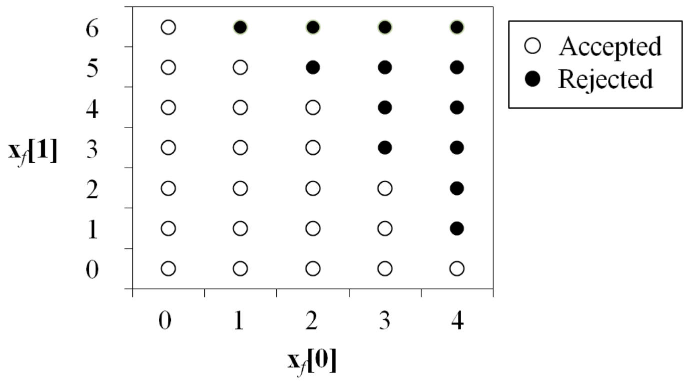
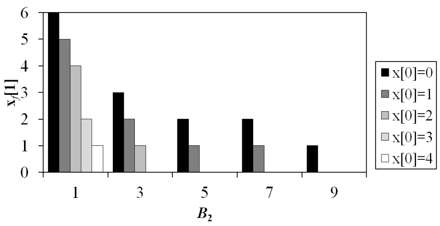
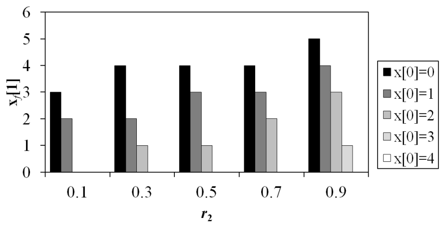


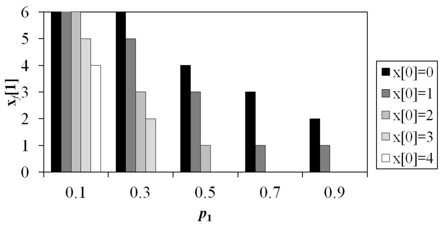
Figure 4.8: *Acceptance/Rejection Policy of the FAH with Realistic Scenario for each Aggregated State.* $L_1 = 4$, $L_2 = 11$, $p_1 = 0.1$, $p_2 = 0.5$, $r_2 = 0.1$, $B_1 = 1$ *and* $B_2 = 2$.



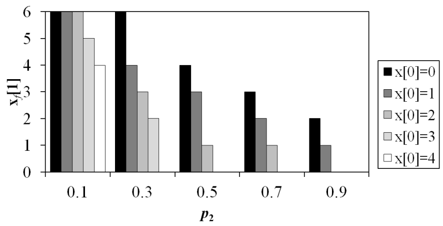
(a) Sensitivity to B_2. $p_1 = 0.5$, $p_2 = 0.5$, $r_2 = 0.5$.



(b) Sensitivity to r_2. $p_1 = 0.5$, $p_2 = 0.5$, $B_2 = 2$.



(c) Sensitivity to p_1. $p_2 = 0.5$, $r_2 = 0.5$, $B_2 = 2$.



(d) Sensitivity to p_2. $p_1 = 0.5$, $r_2 = 0.5$, $B_2 = 2$.

Figure 4.9: *Threshold Values of the FAH with Realistic Scenario.* $L_1 = 4$, $L_2 = 11$ *and* $B_1 = 1$.

4.6.4 Efficiency of the PAH

Despite the excellent overall performance of the FAH with the realistic scenario, the optimality gap remains significant for 9.76% of the instances generated ($Gap_r(0) \geq 1\%$), as we observe in Figure 4.6. For these instances, we study how the systematic disaggregation of information related to already accepted orders can improve the quality of the acceptance policies found. For this, we calculate $Gap_r(z)$ for $z \in \{1,2,\ldots,6\}$. The average values of $Gap_r(z)$ are displayed in Figure 4.10. The optimality gap of the PAH with the realistic scenario decreases rapidly as the value of z grows. In particular, we note from Figure 4.10 that when z increases from 0 to 4, the average gap decreases from 1.54% to 0.73% when $L_1 = 3$ and from 2.05% to 0.17% when $L_1 = 7$. Thus, the optimality gap of the FAH is greatly reduced while the computational time remains small (see Table 4.2). We remark that for the remaining 90.24% of the instances studied, the average optimality gap is also improved when the value of z increases.

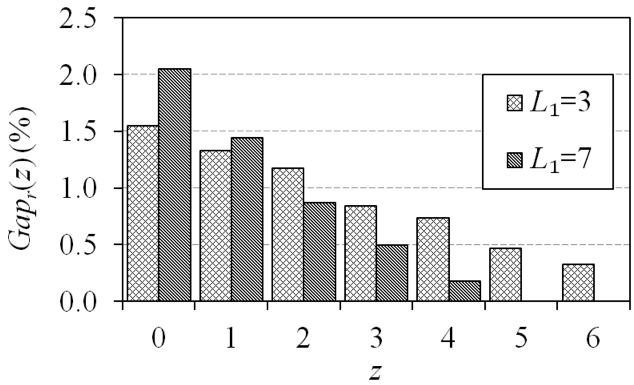


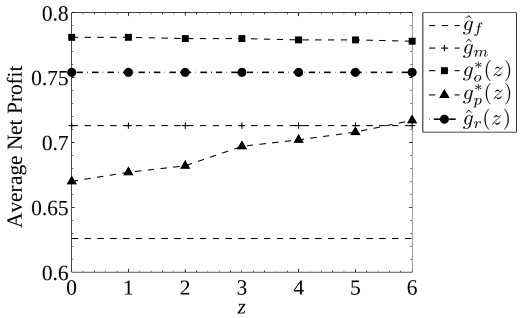
Figure 4.10: *Average Optimality Gap of the Acceptance Policies Under the Realistic Scenario for the Instances with $Gap_r(0) \geq 1\%$ for Different Values of z and L_1.*

4.6.5 Efficiency of the PAH and FAH for Large Lead Times

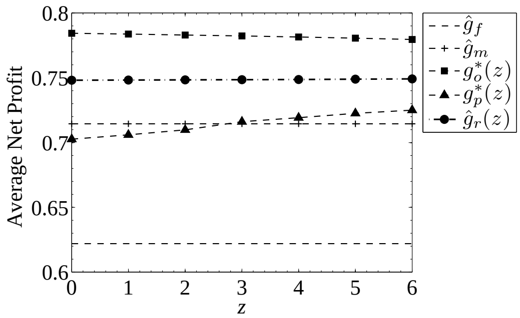
As shown in Table 4.2 an advantage of the FAH and the PAH is to construct the policies in tractable time even for instances for which the optimal steady-state policies cannot be obtained. In order to evaluate the efficiency of the studied heuristics for these instances, we fix the value of L_2 to be large i.e. $L_2 = 29$ while the other parameters take the same values as in Section 4.6.3. As a result our experiments include again 984 instances. We also compute the upper and lower bounds obtained from Proposition 4.1.

The average net profits of the different policies are plotted as a function of z in Figure 4.11a and Figure 4.11b for $L_1 = 3$ and $L_1 = 7$, respectively. We observe the following: first, the FAH and the PAH with the realistic scenario significantly outperform the PLB policy. Second, the effect of increasing z seems much stronger on the quality of the bounds than on the performance of the realistic policy. For a vast majority of instances the disaggregation does not bring any significant improvement.

However, in a very similar fashion to Section 4.6.4, we observe that for the 10% of instances with the worst performance for the FAH, the profit increased by at least 0.5% between the PAH heuristic with $z = 0$ and $z = 6$. The average improvement for these instances is 0.781%. This seems to indicate that the main usefulness of the PAH is to give the possibility of controlling the quality of the proposed solution.



(a) $L_1 = 3$.



(b) $L_1 = 7$.

Figure 4.11: *Effect of Disaggregation of the Information on the Average Long-Run Net Profits Achieved with Different Policies for the Large Lead Time of Regular Orders.*

4.7 Operational Flexibility

In the previous section, we illustrated the efficiency of the proposed algorithms for a large set of instances. Here, we try to gain some further insight into the circumstances where revenue management would have the most significant impact. We will focus on instances following the pattern of the cases that motivated our work (namely, a class of urgent orders with relatively low demand and a class of regular orders with longer lead times and lower revenues).

As expected, the benefit of revenue management is strongly correlated with the intensity of demand as shown in Figure 4.12. We observe that the potential benefit of revenue management compared to the FCFS policy increases with the difference between L_2 and B_2. This means that the advantage of revenue management is larger when there is more flexibility in processing regular orders. Note that, the effect of the flexibility for urgent orders is more limited.

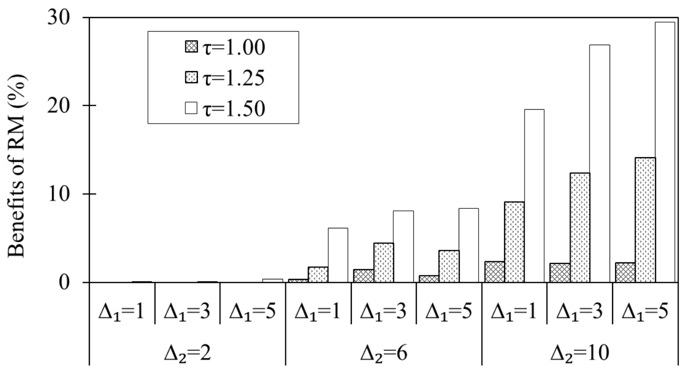


Figure 4.12: *Benefits of the Proposed Revenue Management Approach (RM) Compared to the FCFS Policy for Different Characteristics of the Demand Streams. Results for* $L_1 = 6$, $L_2 = 12$, $\rho = 0.50$ *and* $\beta = 0.50$. $\Delta_i = L_i - B_i$ *where* $i = 1,2$. *The Benefits of RM are Computed by* $(g^* - \hat{g}_f)/\hat{g}_f \times 100\%$.

In order to gain more insights into the impact of revenue management on the operations we compare the acceptance rates of both classes with and without revenue management. Table 4.4 shows the proportion of accepted orders for each demand class. The results illustrate how the revenue management technique is giving gradually higher priority to the urgent orders when ρ (the relative margin of regular orders) is decreasing. We also observe again that the operational flexibility once an order is accepted plays an important role, in this experiment when $B_2 = 10$ the impact of revenue management is minimal, and more generally the smaller B_2 (for a fixed value of L_2) the larger the impact. Of particular interest is the case where $B_1 = 5$ and $B_2 = 10$, that is there is hardly any flexibility for both classes. In that case we observe that for $\rho = 0.75$ or 0.5 the optimal policy is essentially FCFS, while for $\rho = 0.25$ the policy is to accept only the high revenue class. In other words, given the small amount of operational flexibility there is an abrupt switch in the acceptance policy from accept all orders (whenever feasible) to accept only high margin urgent orders when the difference in margin is high enough.

Table 4.4: *Acceptance Rate Under the FCFS and the Optimal Policy and the Benefits of RM for Different Order Sizes and Revenues.*

		FCFS		Optimal, $\rho = 0.75$			Optimal, $\rho = 0.50$			Optimal, $\rho = 0.25$		
B_1	B_2	μ_1	μ_2	μ_1	μ_2	b	μ_1	μ_2	b	μ_1	μ_2	b
1	2	55.6%	91.8%	96.6%	70.8%	4.9%	99.3%	68.9%	14.1%	99.9%	67.7%	31.8%
1	6	77.1%	70.7%	85.1%	65.8%	0.3%	93.4%	59.8%	3.6%	97.6%	55.2%	11.4%
1	10	85.0%	52.4%	85.1%	52.3%	0.0%	85.1%	52.3%	0.0%	90.2%	45.6%	1.6%
3	2	44.8%	95.4%	80.4%	75.2%	2.8%	87.3%	70.3%	12.5%	92.4%	62.3%	33.6%
3	6	57.7%	74.2%	57.8%	74.2%	0.0%	80.5%	57.4%	4.4%	88.5%	45.6%	17.5%
3	10	60.4%	56.5%	60.4%	56.5%	0.0%	60.4%	56.5%	0.1%	97.1%	0.0%	9.6%
5	2	36.1%	97.1%	66.0%	79.3%	1.8%	66.0%	79.3%	9.1%	71.9%	70.1%	26.4%
5	6	40.4%	77.1%	40.4%	77.1%	0.0%	53.4%	66.1%	1.8%	71.0%	48.4%	20.6%
5	10	44.4%	56.8%	44.4%	56.8%	0.0%	44.4%	56.8%	0.0%	79.6%	0.0%	9.3%

Results for $L_1 = 6$, $L_2 = 12$, $\tau = 1.25$ and $\beta = 0.50$. Note that, μ_i is the service level of demand stream i and b is the benefit of RM compared to the FCFS.

4.8 Summary and Conclusions

In this Chapter we studied the order acceptance problem for a firm serving two classes of demand that differ in net profit and lead time over an infinite horizon. We obtained the optimal order acceptance policy by formulating the problem as a multi-dimensional Markov Decision Process. However the resolution for the optimal policy can require high computational requirements. To overcome this difficulty, we proposed an efficient heuristic consisting in a parametric aggregation of the state space. The parameter makes it possible to find the best trade-off between the computational time and the quality of the solution. We propose several variants based on different assumptions about the dynamics of the aggregate state. The variant which assumes that all reserved capacity is distributed uniformly in the aggregation interval gives significantly better solutions than the other approaches studied in the extant literature. The other variants give lower and upper bounds that make it possible to obtain a guarantee about the quality of the solution and the possible gap with respect to the optimal solution. Finally, the computational time remains very short even for instances with large order lead times.

We showed that the amount of operational flexibility (in our case this means how much slack there is between the promised lead time and the effective processing time needed) has a large impact on the performance of revenue management. The more flexibility there is the larger the potential benefit of implementing a revenue management based order acceptance policy. The study of the potential benefits of revenue management associated with flexibility in other contexts is an interesting avenue for further research. In the airline industry for example, the hub and spoke organization is widespread. The major airlines have several hubs and for a journey that is not starting or ending in a hub, it might be interesting to keep some flexibility in terms of the legs traveled by a passenger (i.e. through which hub) as long as a time-slot is respected for the departure and arrival times at the origin and destination respectively. It would be interesting to investigate how the state aggregation policy presented in this article could be extended for the more general network structure of airline operations.

Another direction for future work is the exploration of some properties of the value functions in the MDP formulation. Based on the concept of $L^\natural$-convexity, some recent works (see e.g. Zipkin, 2008, Pang et al., 2012) partially characterize the optimal policies for inventory problems with lead time issues. We conjecture that similar properties exist in the context of our problem. If so, it might be possible to develop more efficient computational methods.

Chapter 5

Conclusions

In this thesis we model analytically the linkage of a firm with its own supply chain (SC) and with firms belonging to other supply chains (SCs). We classify the linkages between entities belonging to the same supply chain as vertical relationships, and the linkages between entities belonging to different supply chains as horizontal relationships. When planning operations taking into account vertical and horizontal relationships, a firm can get significant benefits (cost and risk reduction, increasing profits, capacity utilization, production flexibility, etc.), however, decision makers face new challenges associated to modeling operations. We propose quantitative approaches for planning operations that combine Microeconomic models (bargaining models, game theory, profit maximization), for representing the linkages between firms, and Operations Management techniques (Mixed Integer Linear Programming (MILP), Revenue Management, Markov decision process (MDP), etc.) for optimizing production activities. In this chapter, we discuss theoretical and practical implications of this thesis and future research directions.

5.1 Implications

We analyze the implications of the planning the operations of the firms adequately when considering different types of relationships in SCs.

When a horizontal relationship represents a collaborative alliance between firms, an opportunity arises for improving the performance of such firms. Nevertheless, a collaborative alliance may impose that firms compete for using shared resources, so the objectives of the firms create opposition between them. Thus, implementing collaborative initiatives should be successful in addressing two dimensions of the problem: the first dimension is related to the global performance of the partners, however, the globally optimal solution may not be enough to satisfy the individual expectations; the second dimension is related to the perception of fairness that partners have about the results of the alliance. Indeed, the fairness dimension of the problem is the crux of the sustainability of the agreement through time, because it determines the willingness of the firms to continue working together. We suggest that designing collaborative alliances should be based on two principles: (i) the performance of the collaboration should be measured in terms of the unit costs for the firms and (ii) firms can achieve a good balance between global optimization and fairness by minimizing the maximum unit costs of the firms. As we show in our results, the implementation of these principles may also have a significant impact on the risk associated to the collaboration.

From a managerial point of view, the opportunities arising from horizontal collaboration can lead to competitive advantages for the partners. This may be key for the success of firms in the future, when more competitive environments will demand higher efficiency in utilizing resources. In particular, medium and small-size firms can gain from collaboration initiatives, because by working together they can compete in better conditions with larger firms, so that they can reach similar economies of scale, capacity flexibility, coverage, etc.

Even when the firms do not establish collaborations between each other, other horizontal relationships impose that the internal actions of a firm are highly dependent on the decisions made by others. Such is the case of horizontal competition. In such case the challenge for a firm is to coordinate its internal decisions, such as planning operations and pricing, in order to take into account external influences (opponents' prices) and make its products more competitive in the market. Our numerical results show that firms can get higher profits when facing horizontal competition by utilizing a dynamic pricing strategy. Such strategy helps to control the amount of demand in each production period such that a firm will have the available capacity to satisfy demand. Also, we state that the Operations Department should debate its decisions with other bodies of the firm. For example, by increasing the production capacity, a firm may be hurt because the reduction of operational costs associated to such action may impose a higher competition with its opponent, so the the equilibrium prices may reduce the incomes of the firm.

The models and insights that we propose for horizontal competitions may constitute a guide for decision makers. This idea becomes stronger because the number of firms in different markets and industries grows constantly. Even though the problem of competition between firms is latent since many years, only a few works focuses on including operations that requires more complex computations. Thus, our work is getting closer to practice and gives a strong motivation for the implementation of practical decision aid tools taking into account theoretical models of competition and well-known production-inventory problems. Thus, firms can utilize our work as a tool for competing adequately in their markets, and also, to coordinate the decisions of its Marketing department and Operations department.

In addition to the horizontal relationships, we also study the influence of the characteristics of a SC on the planning of operations of a firm. In this frame, we analyze how a firm should selectively accept customer orders and schedule its production in order to benefit from the different characteristics of customers downstream in the same SC. This constitutes an example of a vertical relationship in a SC. We propose a methodology that selects customer orders based on the profit per unit of accepting an order and the resources required for satisfying such order. So, the firms can maximize their profits by setting a policy that discerns whether to accept or to reject an order.

Given the stochasticity of the demand of the customers, we model the dynamics in the operations of the firm as a Markov Decision Process (MDP). Although a firm can utilize this formulation to set an acceptance policy that maximizes its expected profits, such formulation may become intensive in consuming computational resources. We propose aggregation heuristics that simplify the space of our MDP representation by considering partially the information of the due date of the already accepted orders. Through making simple assumptions about the omitted information we propose highly efficient policies, which are obtained in a shorter time. Further, by numerical results we show that there is a link between the acceptance policy of the firm and the capacity tightness.

Our analysis of vertical relationships in a SC can be useful for decision makers. Given that the customer are becoming more demanding, the firms should plan operations in order to be able to satisfy the customer requests. But the existence of more competitive environments imposes that firms go further than the demand satisfaction. The firms should be able to meet their promises to customers, in such a way that they can build long term vertical relationships. In that sense, our methodology can be utilized to ensure lead time promised to customers while maximizing the profits of a firm. Also, our approach can be utilized in real applications different to manufacturing. For example, airlines could consider our approach for offering flexible flights to customers, so the airline will respect the time-slot for the departure and arrival time at the origin and destination, but the legs traveled by the passenger can be modified.

5.2 Research Directions

In addition to the future research proposals already described in Sections 2.7 , 3.7 and 4.8 , here we discuss new avenues for research that combine the different topics analyzed in this thesis.

5.2.1 Horizontal collaboration and horizontal competition

Given that horizontal collaborations become attractive when there are similarities in the operations of the partners, we could assume that partners offer products that compete for the same customers. Thus, we propose to study how to combine the methodologies for addressing horizontal competition and horizontal collaboration. In addition to the challenges discussed in this thesis, the firms will have to face new theoretical and practical issues. From a theoretical point of view, jointly planning of operations of the firms while computing the pricing equilibrium constitutes a problem whose solution will involve a high computational effort. So, improving the computational time of solving the joint operations planning may be the crux of the methodology. From a managerial point of view, any approach dealing with simultaneous collaboration and competition should address the legal constraints regarding collusion practices, e.g. the agreement cannot reduce the welfare of the customers.

5.2.2 Horizontal collaboration and vertical relationships

Manufacturers that selectively choose orders from different demand streams may also discuss the possibility of a collaborative alliance with similar firms belonging to other SCs. So, an interesting extension of our work consists in including horizontal collaborations in the problem of accepting/rejecting orders. Researchers could consider collaborations in which two or more manufacturers pool their production lines; hence the challenge for the manufacturers is to agree on a scheme of collaboration by which they jointly decide when the production can be transferred from one production line to another one. Implementing this type of initiative will lead to higher rates of demand satisfaction due to the flexibility of using exchangeable production lines of the partners. The computation requirements for designing an acceptance policy will be high, because both the size of the instance will grow with the number of partners and fairness constraints should be incorporated in the models. Thus, our approximation heuristics will be essential to decrease the computational complexity.

5.2.3 Horizontal competition and vertical relationships

We suggest investigating the effects of horizontal competition on the acceptance/rejection decisions problem. Here, we could expect that the pricing and lead time decisions of a set of manufacturers may influence the demand of each other. Thus, the challenge for a manufacturer is to offer a combination of prices and lead times to the retailers that allow him to compete with his opponents. Again, the complexity of computing an equilibrium between manufacturers may be high, so the proposed heuristics for acceptance/rejection decisions could be utilized as a subroutine required for computing a market equilibrium.

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